Applied Algebra II

Doug Gardner
Rogue Community College

Rogue Community College
3345 Redwood Highway
Grants Pass, OR 97527-9298
www.roguecc.edu
(541) 956-7500

An Innovative Math in CTE Curriculum:

Funded by an ATE grant from the National Science Foundation:
Principal Investigators:
Doug Gardner
Serena Ota St. Clair

Chapter Objectives

Chapter 1: Linear Relationships

Section 1.1: The Shape of a Linear Equation
- Graphing basics
- Graphing ordered pairs
- The meaning of slope
- Trend line

Section 1.2: Finding Linear Equations
- x and y intercepts
- Standard form
- From ordered pairs
- Choosing a trend line

Section 1.3: Using Linear Equations
- Regression
- Solving for x and y
- Modeling

Chapter 2: Quadratic Relationships

Section 2.1: The Shape of a Quadratic Equation
- Vertex
- x and y intercepts
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Section 2.2: Finding Quadratic Equations
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Section 2.3: Using Quadratic Equations
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Chapter 3: Power Relationships

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Section 3.2: Finding Power Equations
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Section 3.3: Using Power Equations
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Chapter 4: Exponential Relationships

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Chapter 5: Logarithmic Relationships

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Section 5.2: Finding Logarithmic Equations
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- Solving for y: order of operations
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Section 5.3: Using Logarithmic Equations
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Chapter 6: Choosing the Right Model

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Section 6.2: Resistant Data
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Appendix: Problem Citations

Introduction:

This book is designed to be read. Some humor and stories of my personal journey in understanding and applying the beautiful language of mathematics are included to keep it interesting. There are no shortcuts, however! If you are to understand the material deeply enough to apply it to all the rich problems that life presents, you will need to add a considerable amount of your own thought and dedication to the process. Your goal should be to "own" the material rather than merely "borrow" it. Owning it means that you carry the knowledge with you, like a carpenter carries his or her tools, ready to use them as needed. Ownership, however, comes at a cost, and the amount you must pay is dependent on your natural ability, interest, and the time and energy you are willing to put into it. Remember, if it was as easy as digging a ditch, everyone could do it and you would not get paid extra for knowing it.

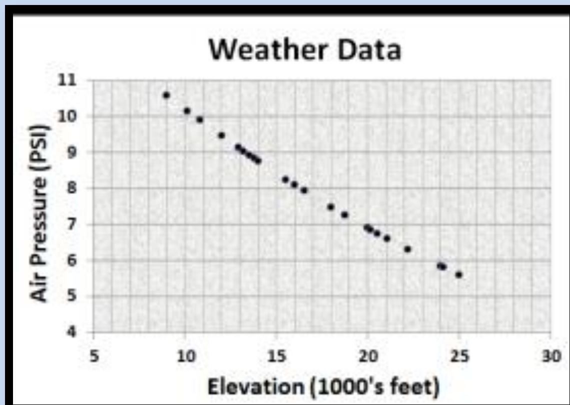
Algebra is, ultimately, a useful method for expressing relationships that are numerical. A numerical relationship can be expressed as **ordered pairs**, as a **graph** or with an **equation**.

For example, there is a relationship between elevation and air pressure. As you climb higher, the air pressure decreases.

Ordered Pairs - The national weather service sends up weather balloons across the nation every day to collect data on this relationship to forecast the weather. Some data from Medford, Oregon is shown below.

Elevation (x1000 feet)	Air Pressure (PSI)
9.00	10.60
10.15	10.15
10.84	9.89
12.00	9.46
12.92	9.14
13.21	9.04
13.54	8.92
13.79	8.83
14.00	8.76
15.53	8.25
16.00	8.10
16.53	7.93
17.99	7.48
18.77	7.25
20.00	6.90
20.18	6.85
20.54	6.74
21.07	6.60
22.21	6.29
23.97	5.85
24.15	5.80
25.00	5.59

Graph – The graph shows that the relationship has a definite pattern which allows you to make an educated guess for values outside your data. The graph predicts the air pressure to be about 4.4 PSI at an elevation of 30,000 feet.



Equation – The equation $y = 15.304 \cdot (.961)^x$ gives an even more accurate estimate. Entering 15.304•(.961)^30 into your calculator gives an air pressure of 4.64 PSI at 30,000 feet.

The altimeter exploits this relationship so pilots know their elevation … even in the fog. Algebra is a tool, not unlike a skill saw, in that it takes some knowledge and practice to use, but will allow you to build some amazing things.

This book will take you through common algebraic relationships you may encounter in the world and give you confidence in progressing from ordered pairs, to a graph, to an equation. Creation and use of equations is the main purpose of any algebra course since equations give you the most options and are very concise.

Introduction

As motivation for your study, consider the following real estate application:

Example: Tracking Home Value

Consider the value of a family home for each of the first 6 years of ownership:

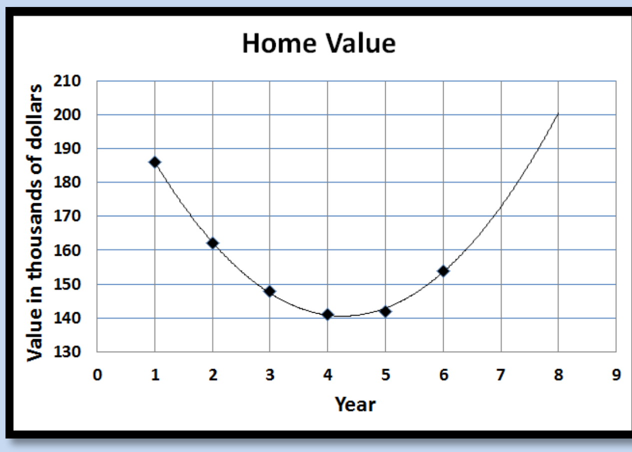
Year	Home Value ($ in thousands)
1	186
2	162
3	148
4	141
5	142
6	154

The owners were hoping to make a little money on the house. At the present rate, when can they expect the value of the home be to reach $200,000 since the value appears to be back on the rise?

Solution:

First, there appears to be a relationship in the ordered pairs between time and value, in that the value dropped during the first 4 years then began to rise in years 5 and 6.

Second, graphing the relationship with years on the horizontal axis and home values on the vertical axis shows a definite and predictable pattern.



Third, following the pattern in the graph it appears that the value would be predicted to reach $200,000 about year 8.

Finally, we will learn in chapter 2 how to determine an equation that fits the data. For now note that the equation $V = 4.3T^2 - 36.5T + 218$ models the relationship between value (V) and time (T).

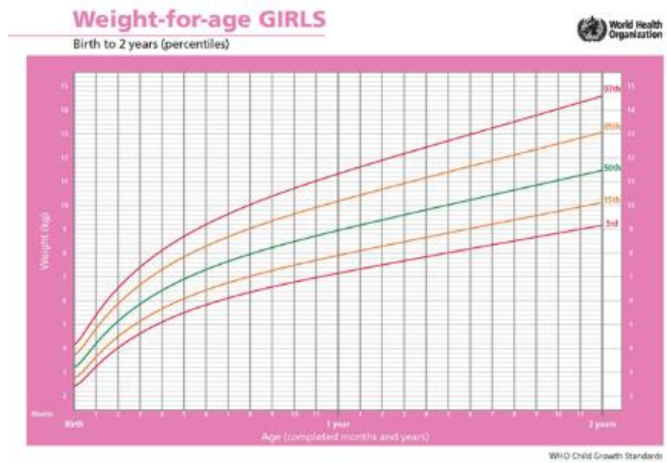
Final Answer: The family can sell the house around year 8 for $200,000.

Note: this prediction assumes the market value continues on this pattern upward. It would not be at all unusual for the housing market to change pattern before then.

The equations modeling elevation -vs- air pressure and time -vs- home value are unfamiliar now, and you may not remember all the details of the order of operations or how to use your calculator skillfully, but appreciate that algebra is a very practical tool.

This course will focus on the study of algebraic relationships that people have used to make real predictions solving real problems. The basic idea is that there are quantities in the world that can be shown to be related by gathering data and noticing a predictable pattern on their graph. There are equations that model all types of curves, and a familiarity with how to create and use equations is a key aspect to the technical aspects of our everyday lives.

Medical technicians can track and advise their patients based on patterns in their health.



City planners can build new schools and roads based on population growth models.

Contractors can predict the necessary beam dimensions to support a 2,400 pound roof load.

Algebra is truly all around us; it is simply the study of relationships. Let's begin in chapter 1 studying linear relationships.

Chapter 1:

Linear Relationships

Section 1.1: The Shape of a Linear Equation

Consider the dosage chart below. If we compare the mean weight to the actual dose given it would appear that the dosage increases with weight, as one might expect. A health care provider might use a chart like this to approximate dosage.

Weight Category	< 38 kg	38-47 kg	48-63 kg	64-77 kg	78-97 kg	98-117 kg	118 + kg
Mean weight	29.6	44.2	56.8	70.6	86.0	105.2	122.9
Mean BSA (DuBois method)	1.045	1.370	1.603	1.818	2.015	2.215	2.371
Mean BSA (Mosteller method)	1.045	1.372	1.613	1.841	2.064	2.313	2.535
Actual Dose given	60ml	70ml	80ml	100ml	120ml	140ml	160ml
If dose were 1.416 ml per kg	41.9ml	62.6ml	80.5ml	100ml	121.8ml	149.0ml	174.1ml
If dose were 55.02ml per m² (BSA using DuBois method)	57.5ml	75.4ml	88.2ml	100ml	110.9ml	121.9ml	130.5ml
If dose were 54.30 ml per m² (BSA using Mosteller method)	56.7ml	74.5ml	87.6ml	100ml	112.1ml	125.6ml	137.7ml

http://www.halls.md/ct/dosedata.htm

Two problems present themselves that skill with algebra will solve:

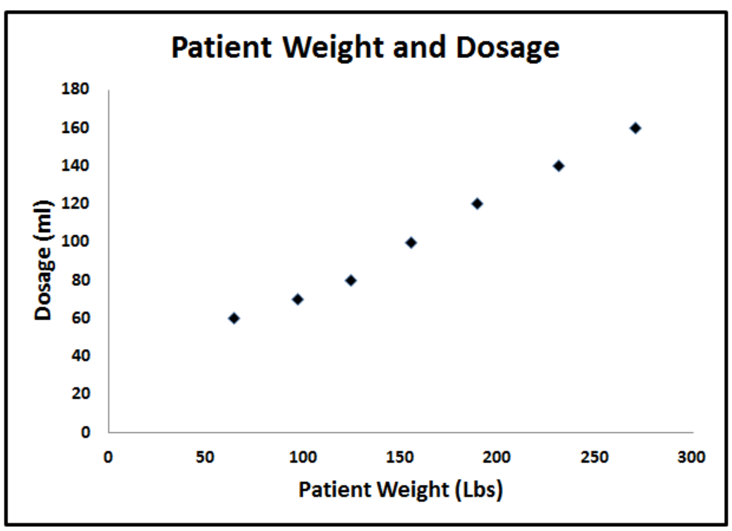
1. What if the patient weight is in pounds instead of kilograms?
2. What if the patient weight is not represented on the chart?

Recall that conversions to other units are easily made applying knowledge of fractions.
29.6 kg would need to be multiplied by a number with units $\frac{lbs}{kg}$ so that kg cancel, leaving our number in pounds. A quick google-search informs us that 1 kg ≈ 2.205 lbs. So, $\frac{29.6\ \cancel{kg}}{1} \times \frac{2.205\ lbs}{1\ \cancel{kg}} \approx 65\ lbs.$
Converting all 7 weights to the nearest pound we find the ordered pairs listed below.

mean weight (lbs)	dose given (ml)
65	60
97	70
125	80
156	100
190	120
232	140
271	160

Note that they are quite logically called "**ordered pairs**".
Pairs, in that it is a pair of numbers that are related.
Ordered, in that order matters (giving a 120 pound patient 190 ml of medicine would be a regrettable mistake).

Ordered pairs are commonly written in the form (x,y). By convention the first number is the **independent** variable and the second number is the **dependent** variable. In this example the dose "depends" on the weight of the patient. When representing ordered pairs on a graph the independent variable (x) is assigned to the horizontal axis and the dependent variable (y) to the vertical axis. The "shape" of the data is approximately a line so we call this a **linear relationship**.



Drawing a "**trend line**" through the points would allow us to solve our second problem and assign a fairly accurate dosage for a weight that is not represented on the chart.

Next consider that the line has **slope**, which turns out to be both interesting and useful.

Slope is defined as the **rise** divided by the **run** so that steeper lines have larger slopes. To find a number for the slope, first choose any two points that represent the line; the rise is the vertical distance between the points and the run is the horizontal distance between the points. The second and the seventh points were chosen in figure 1.1.1. **Important:** *If the trend line does not pass through any of the points, you are free to choose your own.*

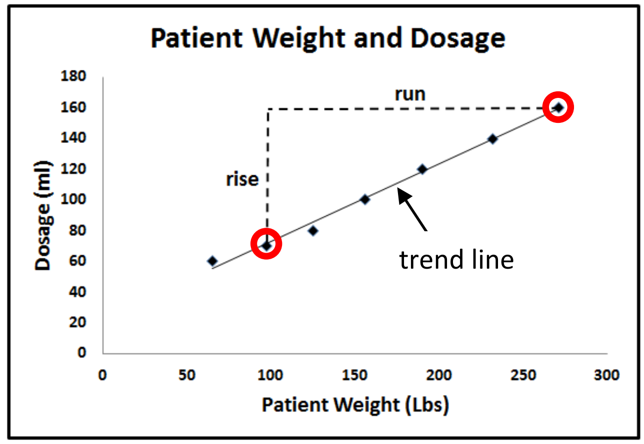


Figure 1.1.1

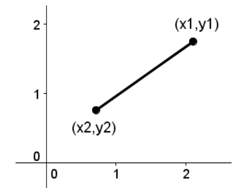
mean weight (lbs)	dose given (ml)
65	60
97	70
125	80
156	100
190	120
232	140
271	160

Rise, in our dosage example, is found by subtracting the dosages (160ml - 70ml = 90ml). Run is found by subtracting the weights (271 lbs - 97 lbs = 174 lbs).

So the slope is $\frac{90}{174} = \frac{15}{29} \approx .52$.

With the units included it has a useful meaning. Since $.52 = \frac{.52\text{ ml}}{1\text{ lb}}$, we can **add** about .5 ml to our dosage for every **increase** of 1 pound. It is also true that we can **subtract** about .5 ml for every **decrease** of 1 pound since $\frac{.52}{1} = \frac{-.52}{-1}$.

Formally then, $\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_1 - y_2}{x_1 - x_2}$; but practically it refers to the <u>rate of change</u> of one quantity relative to another.



It is interesting to note that choosing different ordered pairs will give you a slightly different slope since the seven points are not in a perfect line. Choosing the 1st and the 5th, slope = $\frac{120-60}{190-65} = \frac{60}{125} = \frac{12}{25} = .48$.

Consider a practical slope (rate of change) example involving car value:

Example 1.1.1: Calculating slope

Find and interpret the slope for the car value from the table between the years 1998 & 2005.

year	car value
1998	$27,500
2001	$22,200
2005	$15,700
2007	$12,800
2013	$2,800

Solution:

slope = $\frac{15700-27500}{2005-1998} = \frac{-11800}{7} \approx \frac{-1686}{1}$

Meaning: a **decrease** of approximately $1686 in value for every **increase** of one year in age.

Note: It does not matter which order you subtract as long as your subtraction starts with x and y from the same point. The other option: slope = $\frac{27500-15700}{1998-2005} = \frac{11800}{-7} \approx \frac{1686}{-1}$.

Meaning: an **increase** of $1686 in value for every **decrease** of one year in age. Numerically, the slope is -1686 in either case.

As demonstrated in figure 1.1.1, a useful graph adheres to the following conventions:

1. independent variable (x) on the horizontal axis
2. dependent variable (y) on the vertical axis
3. a title
4. labels (including units) and a scale for each axis

Consider a practical graphing example involving the same car values:

Example 1.1.2: Graphing

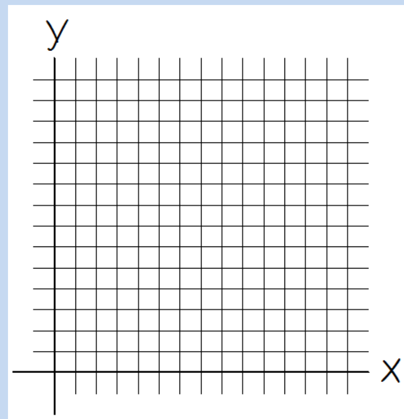
Create a usable graph for the car value from the table. Include a trend line and give a reasonable value for the year 2009.

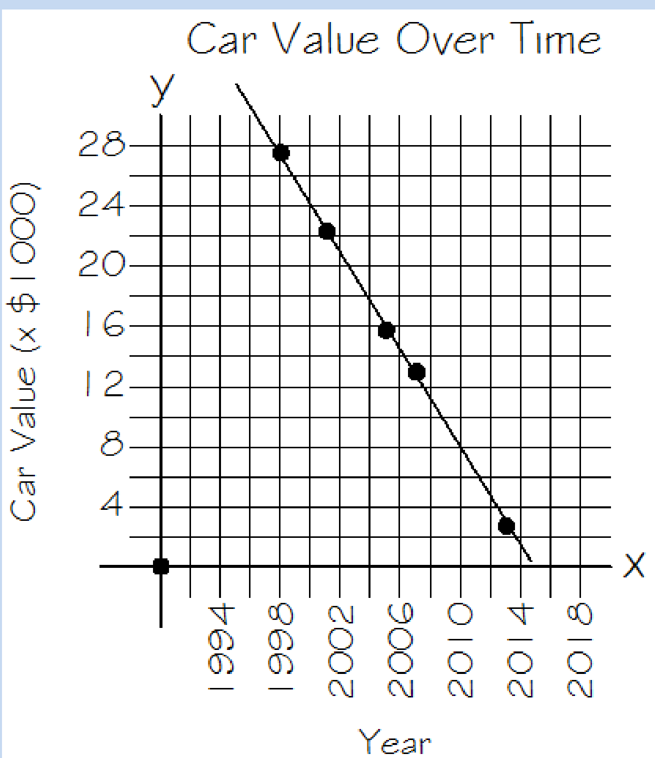
year	car value
1998	$27,500
2001	$22,200
2005	$15,700
2007	$12,800
2013	$2,800

Solution:

(a) Horizontal Axis - Year is the independent variable since the value depends on the year. There are 15 years between 1998 and 2013. Our graph paper does not have 15 lines so the scale will be 2 years.

(b) Vertical Axis - There is $24,700 between $2,800 and $27,500. $24,700 divided by 14 lines is $1764 which can be rounded up to $2000 for a scale that will be easy to read.





We do not know until the graph is done that the shape of the graph is a line. It would not be at all unusual to have found it curved.

Final Answer: Reading the graph, the car looks to be worth a little under $10,000 in 2009.

Note: Choosing the scale for a graph is a dynamic process and there is often more than one right way of doing it.

Note: The x & y **intercepts** are often interesting, in this case the car's value looks to reach zero in 2015 (the point (2015,0) is the x-intercept).

Section 1.1: Problem Set

1. The Kilowatt-hours (KWH) of electricity a home uses each month are dramatically affected by the temperature difference inside versus outside measured in heating degree days (HDD),(*if it is 60 degrees outside and you heat your house to 72 degrees for 5 days that is 12x5=60 degree days*).

Heating Degree Days (HDD)	KWH Used
714	1799
469	1269
386	1160
266	860
133	804
62	696
7	677
5	666
87	736
354	1071
561	1602
968	2060

   a) Make a graph of the data (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
   b) Find the slope between the 133 and 561 degree days, accurate to 2 decimal places.
   c) Add a trend line to the graph.
   d) Choose two representative points from your trend line and find the slope. Explain the meaning of the slope in context.

2. A forester takes measurements in a grove of trees. Naturally larger diameter trees also have larger volumes.

Diameter (cm)	Volume (m^3)
7.2	143
7.4	157
7.5	168
7.7	178
7.8	189
8.0	199
8.2	212
8.5	221
8.7	237
8.8	248
9.0	261
9.3	272
9.4	278
9.6	288
9.8	299
10.1	304
10.3	318
10.4	327
10.6	340
10.9	349

   a) Make a graph of the data large enough to include a 13 cm diameter (use graph paper, label completely, let diameter be the independent(x) variable and volume be the dependent(y) variable).
   b) Find the slope between the smallest and largest diameter trees, accurate to 1 decimal place.
   c) Add a trend line to the graph.
   d) Choose two representative points from your trend line and find the slope. Explain the meaning of the slope in context.
   e) Use your trend line to estimate the volume of a 13 cm diameter tree.

3. A forester takes measurements in a grove of trees. Naturally older trees also have larger volumes.

Age	Volume (m^3)
21	143
22	157
23	168
24	178
25	189
26	199
27	212
28	221
29	237
30	248
31	261
32	272
33	278
34	288
35	299
36	304
37	318
38	327
39	340
40	349

   a) Make a graph of the data larger enough to include a 50 year old tree (use graph paper, label completely, let age be the independent(x) variable and volume be the dependent(y) variable).
   b) Find the slope between the youngest and oldest trees, accurate to 2 decimal places.
   c) Add a trend line to the graph.
   d) Choose two representative points from your trend line and find the slope. Explain the meaning of the slope in context.
   e) Use your trend line to estimate the volume of a 50 year old tree.

4. A study conducted in the early 1960's in 9 counties in northern Oregon and southern Washington related exposure to the Hanford nuclear plant and mortality rate. Exposure is an index number assigned to a county based on its population and proximity to the contaminated water. Mortality is the rate of deaths per 100,000 man-years (*6 people living for 70 years is 6x70=420 man-years*).

Exposure	Mortality
2.5	147.1
2.6	130.1
3.4	129.9
1.3	113.5
1.6	137.5
3.8	162.3
11.6	207.5
6.4	177.9
8.3	210.3

   a) Make a graph of the data (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
   b) Find the slope between the lowest and highest exposure counties, accurate to 2 decimal places.
   c) Add a trend line to the graph.
   d) Choose two representative points from your trend line and find the slope. Explain the meaning of the slope in context.
   e) Estimate the y-intercept and explain its meaning in context.

5. The winning Olympic discus throw is recorded through the 1900's rounded to the nearest foot. It is interesting to note that 1916 and 1940 are missing due to the first and second world wars.

Year	Throw (ft)
1900	118
1904	129
1908	134
1912	148
1920	147
1924	151
1928	155
1932	162
1936	166
1948	173
1952	181
1956	185
1960	194
1964	200
1968	213
1972	211
1976	221
1980	219
1984	219

a) Make a graph of the data large enough to include 1992 (use graph paper and label).
b) Find the slope between 1900 and 1984, accurate to 1 decimal place.
c) Add a trend line to the graph.
d) Use your trend line to predict the winning throw in 1992.
e) Choose two representative points from your trend line and find the slope. Explain the meaning of the slope in context.

6. Consider the price list for rectangular tarps of different sizes.

Dimensions	Area	Price
5' x 7'		$4
6' x 8'		$5
8' x 10'		$8
10' x 12'		$11
12' x 14'		$15
16' x 20'		$26
20' x 24'		

a) Find the area of each tarp in the chart (area of a rectangle is length x width).
b) Make a graph of the area -vs- price data large enough to include the largest area (use graph paper and label).
c) Find the slope between the $5 and $15 tarps.
d) Add a trend line to the graph.
e) Use your trend line to find a price for the 20'x24' tarp.
f) Choose two representative points from your trend line and find the slope. Explain the meaning of the slope in context.

7. Download speeds for data are increasing with technological advances. The speeds are recorded in the table for each month in 2009.

Month	Speed (Kb/s)
1	7028
2	7056
3	7278
4	7866
5	8188
6	8265
7	8355
8	8529
9	8694
10	8844
11	9183
12	9332

   a) Make a graph of the data large enough to include 10,000 Kb/sec. (use graph paper and label)
   b) Find the slope between months 2 and 11, accurate to 1 decimal place.
   c) Add a trend line to the graph.
   d) Use your trend to find the month when the speed will reach 10,000 Kb/s.
   e) Choose two representative points from your trend line and find the slope. Explain the meaning of the slope in context.

8. Upload speeds for data are increasing with technological advances. The speeds are recorded in the table for each year.

Year	Speed (Kb/s)
2008	1029
2009	1728
2010	2207
2011	2705

   a) Make a graph of the data large enough to include 2015 (use graph paper and label).
   b) Find the slope between years 2008 and 2011, accurate to 2 decimal places.
   c) Add a trend line to the graph.
   d) Use your trend line to make a prediction for the upload speed in 2015.

9. Smoking rates for adults have declined over the last couple of decades.

Year	Percent
1990	25.5
1993	25.0
1995	24.7
1997	24.7
1999	23.5
2001	22.8
2002	22.5
2003	21.6
2004	20.9
2005	20.9
2006	20.8
2007	19.8
2008	20.6
2009	20.6
2010	19.3
2011	18.9

   a) Make a graph of the data large enough to include 2020 (use graph paper and label).
   b) Find the slope between years 1990 and 2010.
   c) Add a trend line to the graph.
   d) Choose two representative points from your trend line and find the slope. Explain the meaning of the slope in context.
   e) Use your trend line to make a prediction for the percent of smokers you might expect in 2020.

10. Health costs are steadily rising in the United States.

Year	Expenditure
1995	\$3,748
1996	\$3,900
1997	\$4,055
1998	\$4,236
1999	\$4,450
2000	\$4,703
2001	\$5,052
2002	\$5,453
2003	\$5,989
2004	\$6,349
2005	\$6,728
2006	\$7,107
2007	\$7,482
2008	\$7,760
2009	\$7,990
2010	\$8,233
2011	\$8,608

a) Make a graph of the data large enough to include 2020 (use graph paper and label).
b) Find the slope between years 2000 and 2008.
c) Add a trend line to the graph.
d) Choose two representative points from your trend line and find the slope. Explain the meaning of the slope in context.
e) Use your trend line to make a prediction for the health expenditure you might expect per person in 2020.

11. Smoking rates for students are shown in the table.

Year	Percent
1997	36.4
1999	34.8
2001	28.5
2003	21.9
2005	23.0
2007	20.0
2009	19.5
2011	18.1

a) Make a graph of the data large enough to include 2016 (use graph paper and label).
b) Find the slope between years 2003 and 2011.
c) Add a trend line to the graph.
d) Choose two representative points from your trend line and find the slope. Explain the meaning of the slope in context.
e) Use your trend line to make a prediction for the percent of smokers you might expect in 2016.

Section 1.2: Finding Linear Equations

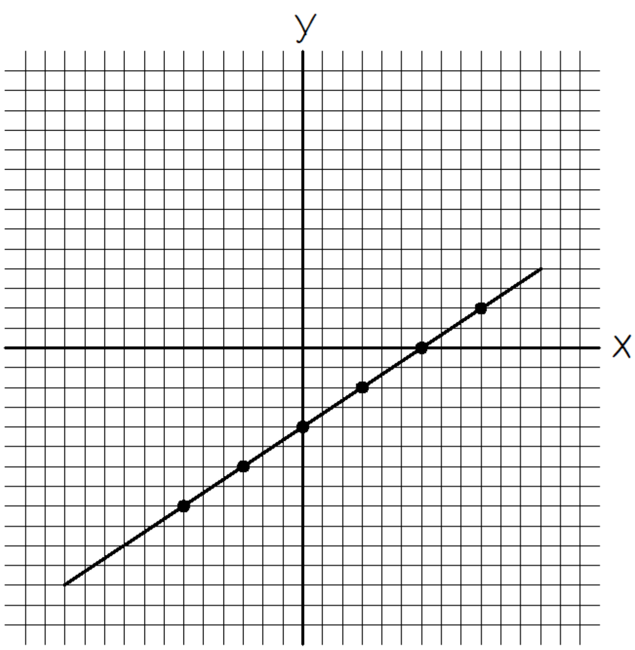
Rene Descartes (1596-1650) is famous for discovering the connection between geometry and algebra. There is an **algebraic equation** that can be found to match the **geometric line** we see in a graph. Indeed there is an equation to match virtually any geometric shape imaginable.

To discover this simple connection, let's start with the equation $y = \frac{2}{3}x - 4$. Solutions to this equation are ordered pairs since there are two variables. The dependent variable is y since it depends on the choice we make for x. We are free to choose any number we like for x.

It will be easy if we pick numbers for x that are multiples of 3 since we are to multiply them by $\frac{2}{3}$. The order of operations dictates that we first multiply our choice for x by $\frac{2}{3}$ then subtract 4. It is very important that you are confident with this process, review the order of operations if necessary. The table shows a partial list of ordered pairs.

x	y
3	-2
6	0
9	2
0	-4
-3	-6
-6	-8

The graph of these ordered pairs is shown below. The algebraic equation produces a geometric line.

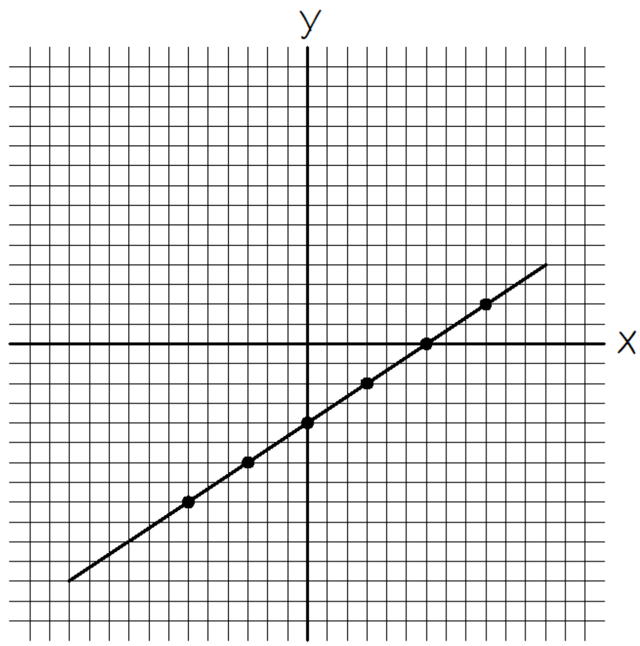


The center of the graph paper (0,0) is named the **origin**.
The **x-coordinate** is located moving right of the origin for a positive x, and left of the origin for a negative x.
The **y-coordinate** is located moving up from the origin for a positive y, and down from the origin for a negative y.

Notice that although 6 ordered pairs are enough to see the pattern, in fact there are infinitely many solutions to the equation since x can be any real number. It is important to understand that the line fills in naturally as you graph more ordered pairs.

The 2-dimensional graph for solutions to 2-variable equations is named the **Cartesian coordinate system** in honor of Rene Descartes (pronounced "day-cart"); a handsome fellow with flowing locks and ahead of his time with a stylish moustache.

Now compare the equation y = $\frac{2}{3}$x – 4 with its graph. It turns out that there is a very simple connection between them. Do you see it?



Hint 1: Find the slope of the line.
Hint 2: Notice where it crosses the y-axis.
The slope is $\frac{2}{3}$ and it crosses the y-axis at -4.
This is more than a fortunate coincidence.
Slope is a rate of change, and the y-intercept occurs where x = 0.

An electric bill is typically the sum of a monthly service fee and a usage fee that is based on the cost of the electricity per kilowatt-hour (KWH). For example, at 5¢/KWH with a $7 monthly service fee, the cost (C) = $\frac{5\ cents}{KWH}$ K + 7 … or
C = .05K + 7 where K is the number of KWH used. 5¢/KWH is the rate and $7 is the cost if K = 0.

The equation of a line is said to be in **slope-intercept form** if it is written in the form y = mx + b. The number in place of m = slope and the number in place of b = y-intercept. Had the English been the first to discover this relationship we might have chosen s for the slope and i for the intercept, but the French beat us to it.

<u>Equation of a line:</u>

y = mx + b

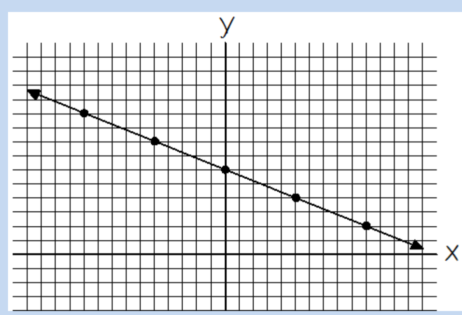
m = slope of the line

b = y-intercept of the line

Consider the following example:

Example 1.2.1: Finding the Equation from a Graph

Find the equation of the line shown on the graph. Assume each square to be 1 unit.



Solution:

We need two pieces of information from the graph … slope and y-intercept:

1. y-intercept = 6 since the graph crosses the y-axis at (0,6)
2. slope = $-\frac{2}{5}$, counting the rise and run between any two points on the graph, (reduce the fraction if necessary).

Final Answer: y = $-\frac{2}{5}$x + 6

Note: It is easy to check your work since the points on the graph should "work" in the equation. Try the point (-10,10): 10 = $-\frac{2}{5}$(-10) + 6 = $\frac{20}{5}$ + 6 = 4 + 6 = 10 … it works!

Consider a practical example involving a rental car:

Example 1.2.2: Finding the Equation from an Ordered Pair

Use the chart to find an equation to model the different costs (C) for a rental car based on the miles (M) it is driven.

miles driven	cost
100	\$57
200	\$82
300	\$107
400	\$132
500	\$157

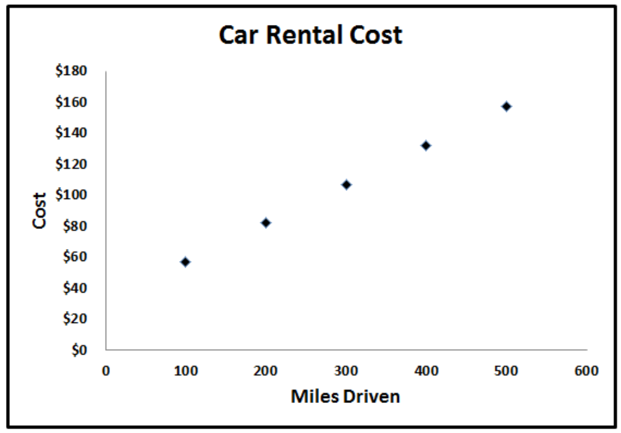
Solution:

The graph of the ordered pairs in the table reveals that the relationship is linear.

slope = $\frac{157-57}{500-100} = \frac{100}{400} = \frac{1}{4}$

We could have chosen any two points to arrive at this slope since the graph is perfectly linear.

So far we have y = $\frac{1}{4}$x + b



From the graph we might guess that the y-intercept is about 40 but we can do better than guess since the ordered pairs must "work" in the equation. Randomly picking the ordered pair (100,57) from the table, and substituting the values in the equation we get:

57 = $\frac{1}{4}$•100 + b

57 = 25 + b

32 = b

Final Answer:

y = $\frac{1}{4}$x + 32 … or better C = $\frac{1}{4}$M + 32 … we now have a formula to find the cost for any number of miles driven.

Note: The slope of $\frac{1}{4}$ indicates that the cost of driving the car is \$1 for every 4 miles driven or 25¢/mile. Remember that slope is $\frac{rise}{run}$ and the units for the rise (y values) are in dollars and units for the run (x values) are in miles.

Reconsider the practical example involving car value where the points do not line up perfectly:

Example 1.2.3: Finding the Equation for a Line

Use the chart to find an equation to model the value for a car purchased in 1996 based on its age, accurate to the nearest whole number.

Consider 1996 to be year zero. *This is not necessary but it will make the process of finding b and its value much more reasonable.*

year	car value
1998	\$27,500
2001	\$22,200
2005	\$15,700
2007	\$12,800
2013	\$2,800

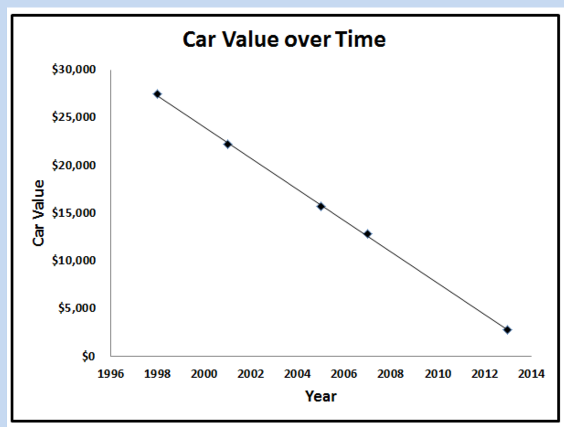
Solution:

The graph reveals that the relationship is close to, though not perfectly, linear. Choose 2 points to calculate the slope that accurately represent the trend line. The first and the last seem a reasonable choice here based on the graph.

slope = $\frac{27500-2800}{2-17} = \frac{24700}{-15} \approx$ -1646.66 VERY IMPORTANT … do not round the slope to the nearest whole number too soon so that b *can be as* accurate as possible. *Storing the number in the calculator may be helpful in minimizing round off errors and saving time.*

As noted in section 1.1 different points will yield slightly different slopes here since this relationship is not perfectly linear.

So far we have y = -1646.66x + b



From the graph we might guess that the y-intercept is about 30,000 but again we can do better since the ordered pairs must "work" in the equation. Let's pick (17,2800).

2800 ≈ -1646.66•17 + b

2800 ≈ -27993.22 + b

30793.22 ≈ b a reasonable result based on the graph!

Final Answer:

y = -1647x + 30793 … or better V = -1647A + 30793 (where V = value and A = age after 1996).

Note: The equation indicates that the car was worth about \$30,800 in 1996 (year 0). The x-intercept indicates the year the car will be worth nothing and we can now calculate it exactly by plugging in 0 for V then solving for A. Businesses establish depreciation schedules for their assets for tax purposes based on the fact that tools and equipment lose value over time.

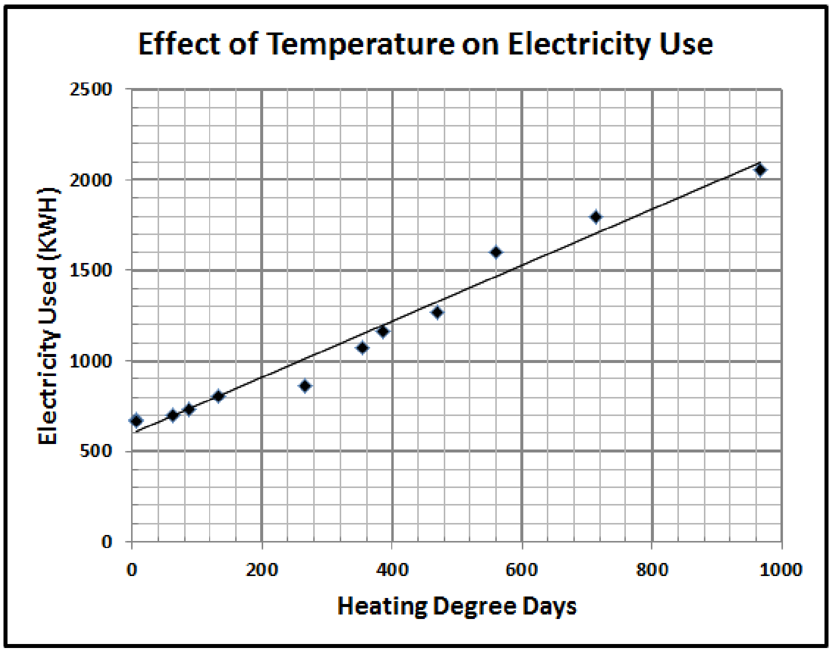
Section 1.2: Problem Set

1. The Kilowatt-hours (KWH) of electricity a home uses each month are dramatically affected by the temperature difference inside versus outside (measured in degree days).

   Round slopes and y-intercepts to 2 decimal places

   a) Find the equation of the line passing through the lowest and highest degree day.
   b) Find the equation of the line passing through the 386 and 62 degrees days.
   c) Find the equation of the line through two representative points from the trend line. The points do not have to be from the data.
   d) Comment on the similarities and differences in the equations you found in a – c.

Heating Degree Days (HDD)	KWH Used
714	1799
469	1269
386	1160
266	860
133	804
62	696
7	677
5	666
87	736
354	1071
561	1602
968	2060

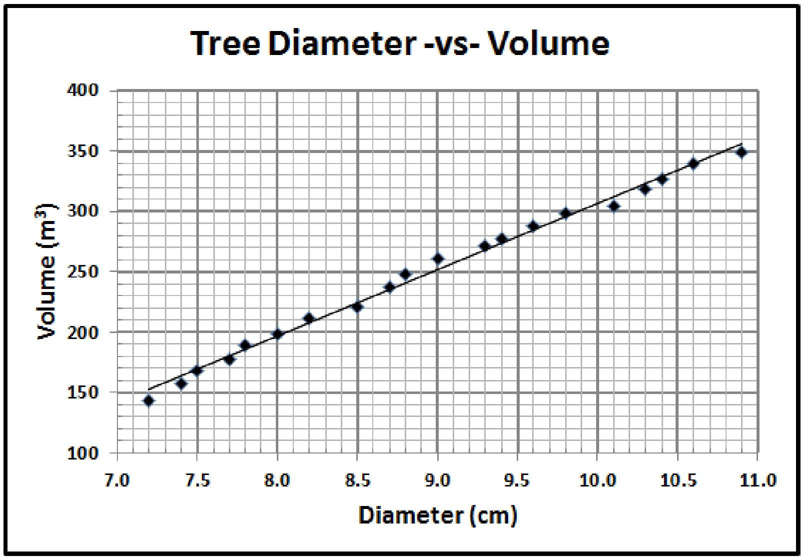


2. Tree diameter and volume are related and of interest to foresters.

   Round slopes and y-intercepts to 2 decimal places

   a) Find the equation of the line passing through diameters 7.2 cm and 8.0 cm.
   b) Find the equation of the line passing through diameters 8.2 cm and 9.8 cm.
   c) Find the equation of the line through two representative points from the trend line.
   d) Comment on the similarities and differences in the equations you found in a – c.

Diameter (cm)	Volume (m^3)
7.2	143
7.4	157
7.5	168
7.7	178
7.8	189
8.0	199
8.2	212
8.5	221
8.7	237
8.8	248
9.0	261
9.3	272
9.4	278
9.6	288
9.8	299
10.1	304
10.3	318
10.4	327
10.6	340
10.9	349

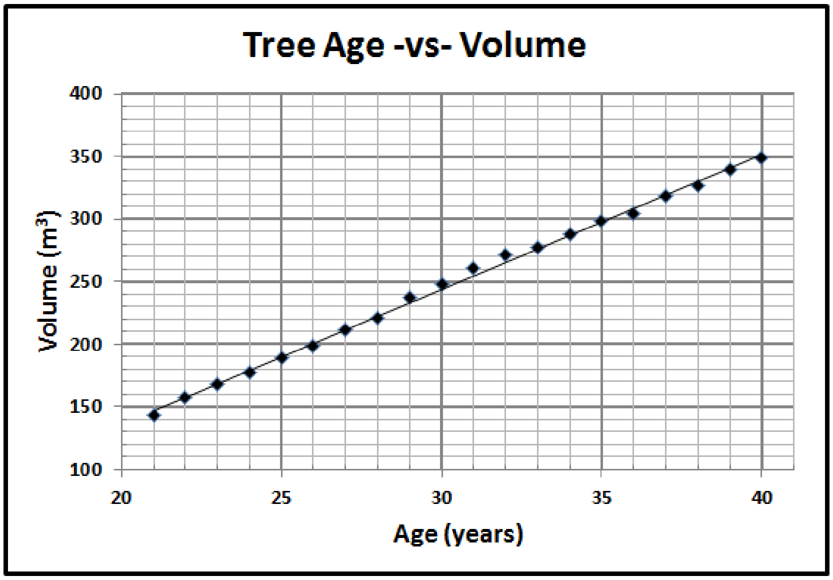


3. A tree's age and its volume are of interest to foresters.

   Round slopes and y-intercepts to 2 decimal places

   a) Find the equation of the line passing through ages 22 and 26 years.
   b) Find the equation of the line passing through ages 30 and 38 years.
   c) Find the equation of the line through two representative points from the trend line.
   d) Comment on the similarities and differences in the equations you found in a – c.

Age	Volume (m^3)
21	143
22	157
23	168
24	178
25	189
26	199
27	212
28	221
29	237
30	248
31	261
32	272
33	278
34	288
35	299
36	304
37	318
38	327
39	340
40	349



4. A study conducted in the early 1960's in 9 counties in northern Oregon and southern Washington related exposure to the Hanford nuclear plant and mortality rate. Exposure is an index number assigned to a county based on its population and proximity to the contaminated water. Mortality is the rate of deaths per 100,000 man-years of life.

Exposure	Mortality
2.5	147.1
2.6	130.1
3.4	129.9
1.3	113.5
1.6	137.5
3.8	162.3
11.6	207.5
6.4	177.9
8.3	210.3

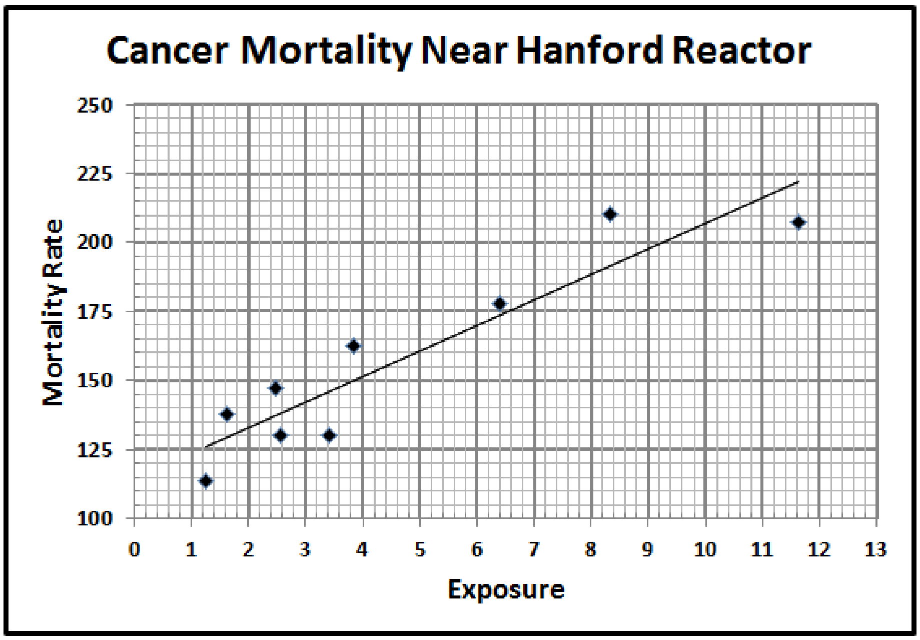
Round slopes and y-intercepts to 2 decimal places

a) Find the equation of the line passing through the lowest and highest exposure counties.

b) Find the equation of the line passing through the 3.4 and 8.3 exposure counties.

c) Find the equation of the line through two representative points from the trend line. The points do not have to be from the data and should not be in this case since the line does not pass through any of them very closely.

d) Comment on the similarities and differences in the equations you found in a – c.

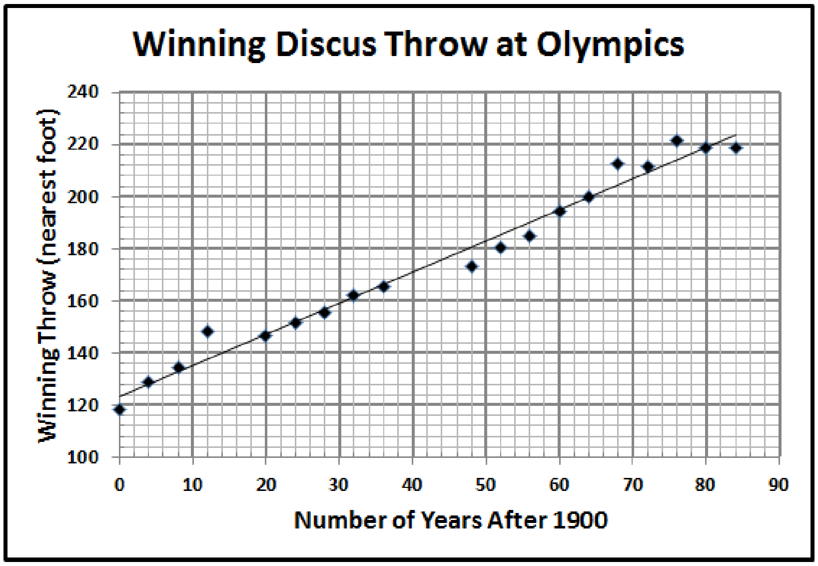


5. The winning Olympic discus throw is recorded through the 1900's rounded to the nearest foot. Consider 1900 to be year 0 for your calculations.

Year	Throw (ft)
1900	118
1904	129
1908	134
1912	148
1920	147
1924	151
1928	155
1932	162
1936	166
1948	173
1952	181
1956	185
1960	194
1964	200
1968	213
1972	211
1976	221
1980	219
1984	219

Round slopes and y-intercepts to 2 decimal places

a) Find the equation of the line passing through the earliest and latest dates.
b) Find the equation of the line passing through the years 1912 & 1948.
c) Find the equation of the line through two representative points from the trend line.
d) Comment on the similarities and differences in the equations you found in a – c.



6. Consider the price list for rectangular tarps of different sizes.

Round slopes and y-intercepts to 3 decimal places

a) Find the area of each tarp in the chart (area of a rectangle is length times width).
b) Use the $5 and $15 tarps to find a linear equation to model the data using area as the x variable and price as the y variable.
c) Use your equation to determine the correct price for the 2 larger tarps in the table to the nearest dollar.

Dimensions	Area	Price
5' x 7'		$4
6' x 8'		$5
8' x 10'		$8
10' x 12'		$11
12' x 14'		$15
16' x 20'		$26
20' x 24'		
24' x 30'		

7. Download speeds for data are increasing with technological advances. The speeds are recorded in the table for each month in 2009.

Month	Speed (Kb/s)
1	7028
2	7056
3	7278
4	7866
5	8188
6	8265
7	8355
8	8529
9	8694
10	8844
11	9183
12	9332

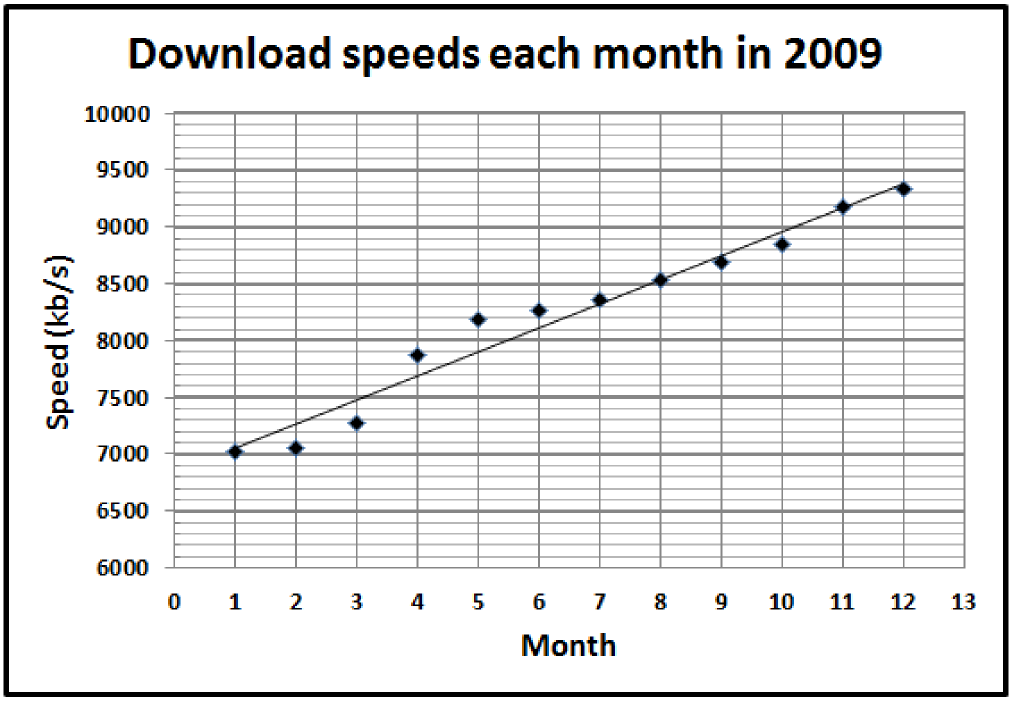
Round slopes and y-intercepts to 2 decimal places

a) Find the equation of the line passing through months 2 & 4.

b) Find the equation of the line passing through months 5 & 7.

c) Find the equation of the line through two representative points from the trend line.

d) Comment on the similarities and differences in the equations you found in a – c.



8. Upload speeds for data are increasing with technological advances. Consider 2000 as year 0.

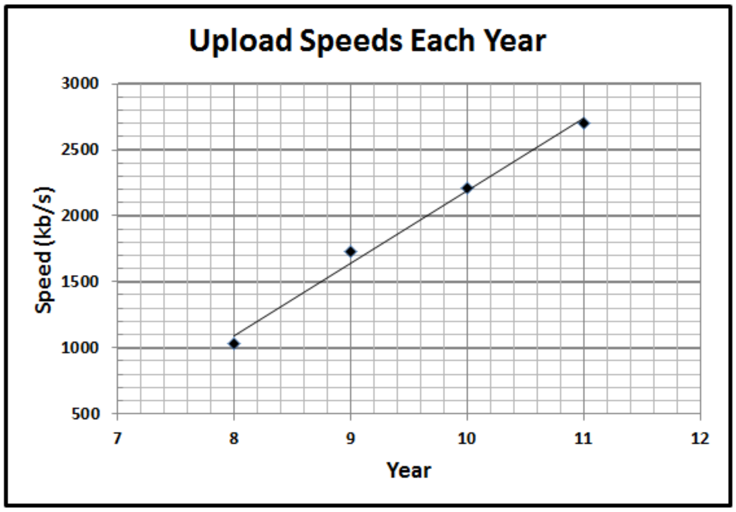
Year	Speed (Kb/s)
2008	1029
2009	1728
2010	2207
2011	2705

Round slopes and y-intercepts to 2 decimal places

a) Find the equation of the line passing through 2008 & 2011.

b) Find the equation of the line passing through years 2009 & 2011.

c) Find the equation of the line through two representative points from the trend line.



9. Smoking rates for adults have been declining over the last couple of decades. Consider 1990 to be year 0.

Year	Percent
1990	25.5
1993	25.0
1995	24.7
1997	24.7
1999	23.5
2001	22.8
2002	22.5
2003	21.6
2004	20.9
2005	20.9
2006	20.8
2007	19.8
2008	20.6
2009	20.6
2010	19.3
2011	18.9

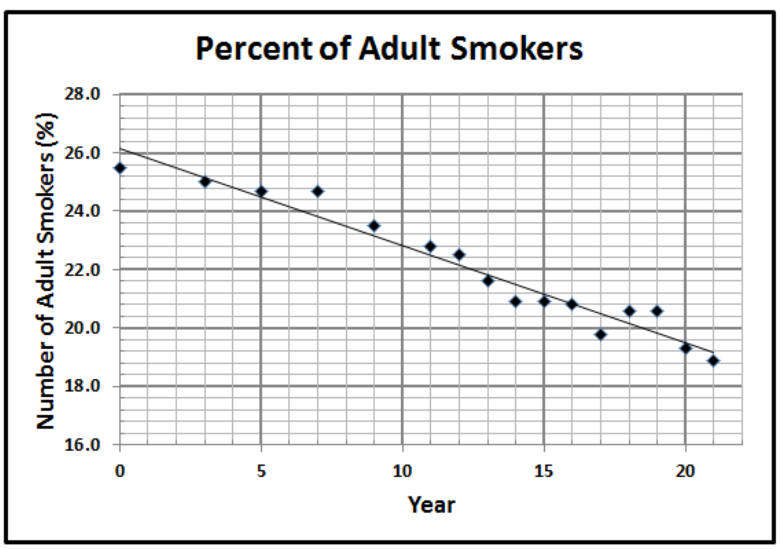
Round slopes and y-intercepts to 2 decimal places

a) Find the equation of the line passing through years 2002 & 2009.

b) Find the equation of the line passing through years 1997 & 1999.

c) Find the equation of the line through two representative points from the trend line.

d) Comment on the similarities and differences in the equations you found in a – c.



10. Health costs are steadily rising in the United States. Consider 1990 as year 0.

Year	Expenditure
1995	$3,748
1996	$3,900
1997	$4,055
1998	$4,236
1999	$4,450
2000	$4,703
2001	$5,052
2002	$5,453
2003	$5,989
2004	$6,349
2005	$6,728
2006	$7,107
2007	$7,482
2008	$7,760
2009	$7,990
2010	$8,233
2011	$8,608

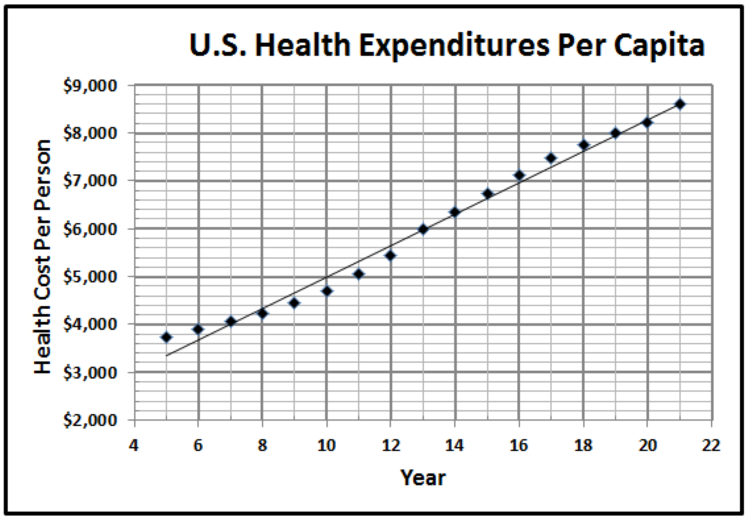
Round slopes and y-intercepts to 2 decimal places

a) Find the equation of the line passing through years 1995 & 1999.

b) Find the equation of the line passing through years 2000 & 2003.

c) Find the equation of the line through two representative points from the trend line.

d) Comment on the similarities and differences in the equations you found in a – c.



11. Smoking rates for students have been declining linearly in recent years as shown in the graph. Consider 1990 as year 0.

Year	Percent
1997	36.4
1999	34.8
2001	28.5
2003	21.9
2005	23.0
2007	20.0
2009	19.5
2011	18.1

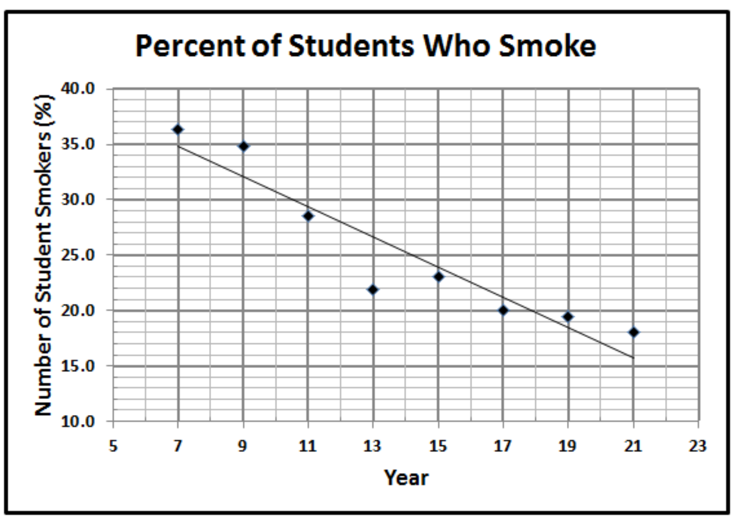
Round slopes and y-intercepts to 2 decimal places

a) Find the equation of the line passing through years 1999 & 2001.

b) Find the equation of the line passing through years 2001 & 2005.

c) Find the equation of the line through two representative points from the trend line.

d) Comment on the similarities and differences in the equations you found in a - c.



Section 1.3: Using Linear Equations

We can now model linear relationships moving comfortably between ordered pairs, graphs and equations. We now come to the real business and purpose of algebra: using our skill to find out something useful that we want to know.

Suppose a doctor is following the health of a patient who is losing weight in order to lower his blood pressure. Consider the chart of his progress:

weight (lbs)	Systolic BP
320	185
290	166
272	154
260	144
252	138

The graph of the data reveals that his progress is close to linear. Find an equation using his heaviest and lightest weight.

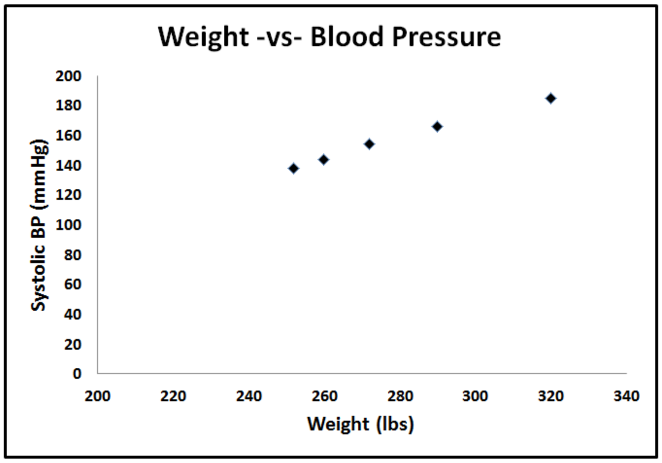
The slope is $\frac{185-138}{320-252} = \frac{47}{68} \approx .69$

Then y = .69x + b

Next, at (252,138) we get 138 = .69(252) + b

b = -35.88

Finally y = .69x – 36 where x is the weight and y is the blood pressure.



So far this is all just mathematical and academic; however, the doctor can now use the equation to predict the weight the patient will need to reach to get his systolic BP down to 120 as the chart below indicates as normal.

120 = .69x - 36	substitute 120 for y
156 = .69x	adding 36 to both sides
x ≈ 226	dividing both sides by .69

The doctor can now tell the patient that a weight of 226 pounds should bring his blood pressure to a normal level of 120.

This chart reflects blood pressure categories defined by the American Heart Association.

Blood Pressure Category	Systolic mm Hg (upper #)		Diastolic mm Hg (lower #)
Normal	less than 120	and	less than 80
Prehypertension	120 – 139	or	80 – 89
High Blood Pressure (Hypertension) Stage 1	140 – 159	or	90 – 99
High Blood Pressure (Hypertension) Stage 2	160 or higher	or	100 or higher
Hypertensive Crisis (Emergency care needed)	Higher than 180	or	Higher than 110

http://www.heart.org/HEARTORG/Conditions/HighBloodPressure/AboutHighBloodPressure/Understanding-Blood-Pressure Readings_UCM_301764_Article.jsp

There are some more convenient ways to look at slope based on how fractions can be written:

Again the slope $\left(\frac{.69\ mmHg}{1\ lb}\right)$ can be very useful, meaning the blood pressure will go **up** about .7 mmHg for every pound **gained**. The slope could also be written $\frac{-.69\ mmHg}{-1\ lb}$, meaning the blood pressure will go **down** about .7 mmHg for every pound **lost**.

Another option can be found dividing both numerator and denominator by -.69, $\frac{-1\ mmHg}{-1.45\ lbs}$, meaning you can **lower** your blood pressure by one "point" for every pound and a half you **lose**.

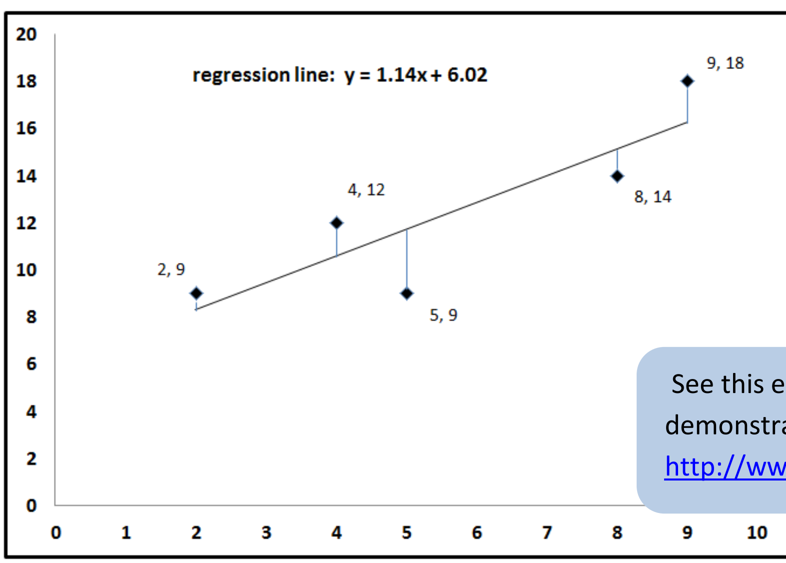
Probably the most useful and memorable version is $\frac{1\ mmHg}{1.45\ lbs} \approx \frac{-2\ mmHg}{-3\ lbs}$. Meaning you can **lower** your blood pressure two "points" for every three pounds you **lose**.

Students often fail to expend the energy to understand algebra stating three basic objections:

1. "When will I ever use this stuff?"
   One can easily avoid algebra in life, but the preceding example illustrates that algebra can be quite useful and help make you more employable.
2. "What do x's and y's have to do with anything?"
   This is a valid observation, perhaps our equation would be more clear if we said B = .69W – 36. The point is that the equation gives a clear mathematical expression that relates weight and blood pressure in a genuinely useful way.
3. "What does .69 times W mean since there is no such thing as math with letters?"
   This is also a valid objection, however because the patient's weight is changing we cannot use a number. The letter is just there to hold the place of a number which, like weight, will change. There is indeed no such thing as math with letters. Letters are simply used in place of a number when the number varies … hence the letters in an equation are called **variables**.

We come now to a technique for finding the equation for the trend line that is both easier and more precise than we have seen thus far.

You may have observed in section 1.1 that each person can place the trend line in a different position, or in section 1.2 they might choose different points to use when finding its equation. We now consider an amazing technique called **least squares regression** that will give us the ***best possible*** trend line. The concept is simple; consider the 5 points in the figure and the vertical distance each is away from the line.



The distances above the line are positive and below the line are negative. The easiest method to find the best trend line is to square the distances to remove the negatives, and then add them to find the least sum possible, hence the name least squares regression.

See this excellent YouTube video for an animated demonstration of least squares regression
http://www.youtube.com/watch?v=jEEJNz0RK4Q

We are indebted to statistics for the ability to find the perfect trend line, or regression line, without all the guessing and adding of distances. In section 1.2 you found the equation for the trend line by considering just two representative points. The amazing thing about the regression line is that it considers **all** the points. If you are beginning to like math (and you should be) you can learn how to find the regression line **by hand** in a statistics class. We are going to use one of the powerful features of the graphing calculator.

See this excellent YouTube video for an animated demonstration of regression on the Ti 83/84 calculator
http://www.youtube.com/watch?v=nw6GOUtC2jY

Everything you need is behind the STAT button on the TI 83/84 calculator.

Example 1.3.1: Using the Graphing Calculator for Regression

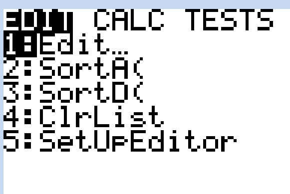
Consider the blood pressure data discussed earlier where we found the equation y = .69x – 36 by using only the first and last point.

Use regression to find the linear equation to model the data.

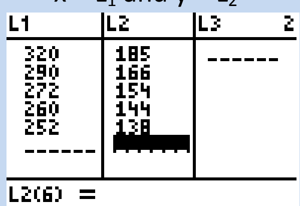
weight (lbs)	Systolic BP
320	185
290	166
272	154
260	144
252	138

Solution:

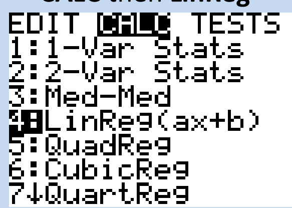
1. Hit the **STAT** button use the **EDIT** menu



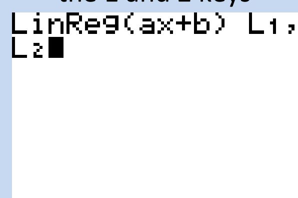
2. Enter the data in **EDIT** $x = L_1$ and $y = L_2$

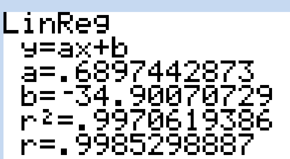


3. Hit the **STAT** button **CALC** then **LinReg**



4. Find L_1 and L_2 above the 1 and 2 keys

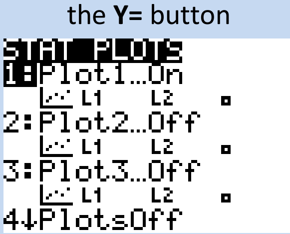




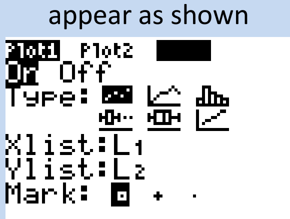
Notice this is close to the equation we found by hand … but much better! The r-value of .9985 tells us it is a good fit since it is very close to 1. It can be "turned on" if you didn't see it on your calculator. Look in the **CATALOG** above the 0 key and scroll down to **DiagnosticOn**.

You can graph the equation and see it passing through the points.

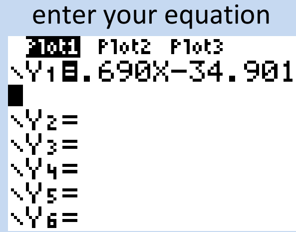
1. Hit **STAT PLOT** above the **Y=** button



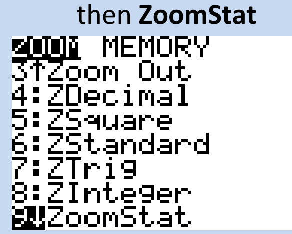
2. Change **Plot 1** to appear as shown

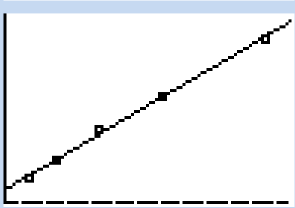


3. Hit the **Y=** button and enter your equation



4. Hit the **ZOOM** button then **ZoomStat**





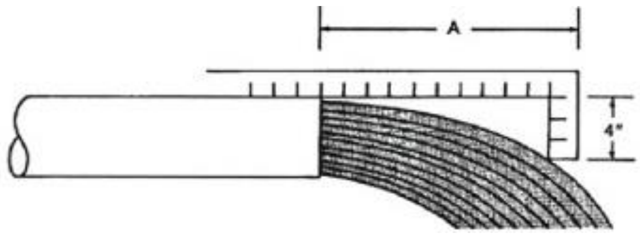
Final Answer:

y = .69x - 34.9 is a good model where x = weight and y = blood pressure.

For the problems in this section we will reconsider the data (ordered pairs) from the previous sections and use TI 83/84 to find the regression line, and then use it to answer predictive questions outside the limits of our data.

Section 1.3: Problem Set

1. The flow rate in gallons per minute (GPM) is shown in the table as a function of the horizontal discharge distance (A) in inches for various pipe diameters.



Round slopes and y-intercepts to 3 decimal places

a) Find the equation for the trend line using the regression feature of your graphing calculator. Consider the flow rate as a function of distance (A) for the 2 inch diameter pipe.

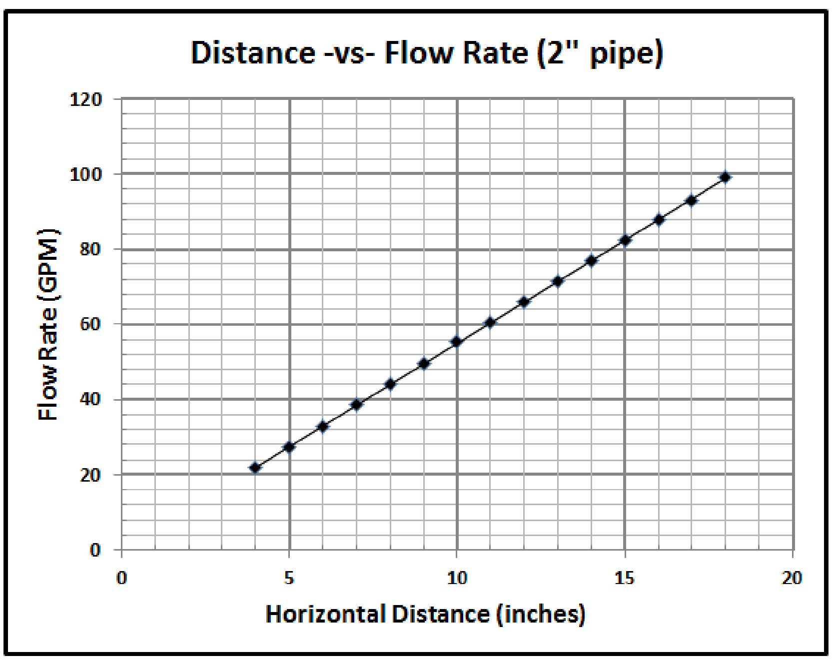
b) Use your equation to predict the flow rate for a horizontal discharge of 24 inches, accurate to 1 decimal place.

c) Use your equation to predict the horizontal distance (A) for a flow rate of 120 GPM, accurate to 1 decimal place.

d) Find the slope between 8 and 10 inches and explain its meaning in context.

Flow Rate (GPM)

Horizontal Distance (A) Inches	1"	1 1/4"	1 1/2"	2"
4	8.70	9.80	13.30	22.00
5	7.10	12.20	16.60	27.50
6	8.50	14.70	20.00	33.00
7	10.00	17.10	23.20	38.50
8	11.30	19.60	26.50	44.00
9	12.80	22.00	29.80	49.50
10	14.20	24.50	33.20	55.50
11	15.60	27.00	36.50	60.50
12	17.00	29.00	40.00	66.00
13	18.20	31.50	43.00	71.50
14	20.00	34.00	46.50	77.00
15	21.30	36.30	50.00	82.50
16	22.70	39.00	53.00	88.00
17		41.50	56.50	93.00
18			60.00	99.00



2. The Kilowatt-hours (KWH) of electricity a home uses each month are dramatically affected by the temperature difference inside versus outside (measured in degree days). The heating degree days are listed in the table from January through December.

Heating Degree Days (HDD)	KWH Used
714	1799
469	1269
386	1160
266	860
133	804
62	696
7	677
5	666
87	736
354	1071
561	1602
968	2060

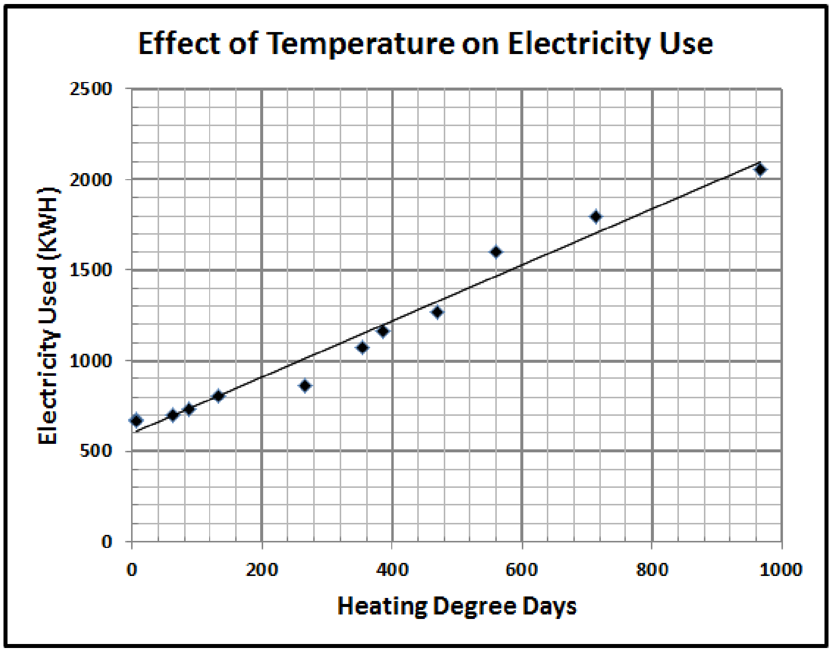
Round slopes and y-intercepts to 3 decimal places

a) Find the equation for the trend line using the regression feature of your graphing calculator.

b) Use your equation to predict the KHW's used for a month with 800 HDD, accurate to 1 decimal place.

c) Use your equation to estimate the HDD for a month that used 2400 KWH, accurate to 1 decimal place.

d) Estimate the y-intercept and explain its meaning in context.



3. Prices for the Java Chip Smoothie at the Human Bean are shown in the table.

Size (oz.)	Price
8	\$3.00
12	\$3.50
16	\$4.00
20	\$4.50

a) Use regression to find the best model for the data.

b) Use your equation to set a fair price for a 32 ounce smoothie.

c) Find the slope between 8 and 12 ounces and explain its meaning in context.

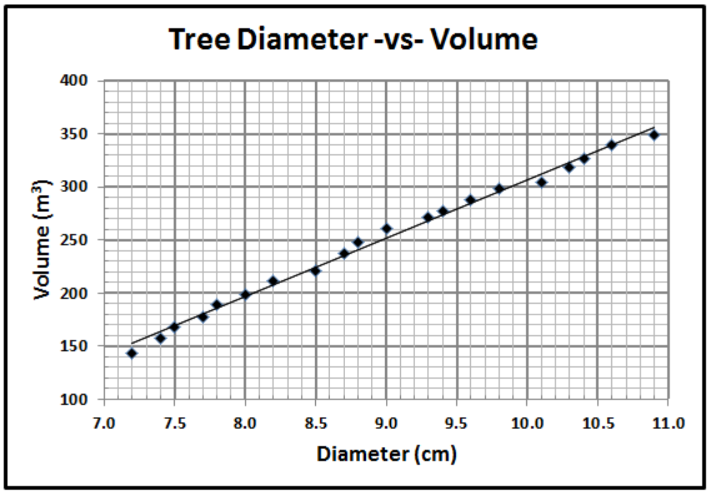
4. A tree's diameter and volume are naturally related.

Round slopes and y-intercepts to 3 decimal places

a) Find the equation for the trend line using the regression feature of your graphing calculator.

b) Use your regression equation to predict the volume of a 5.5 cm diameter tree, accurate to 1 decimal place.

c) Use your regression equation to predict the diameter a tree would need to have a 500 m^3 volume, accurate to 2 decimal places.



Diameter (cm)	Volume (m^3)
7.2	143
7.4	157
7.5	168
7.7	178
7.8	189
8.0	199
8.2	212
8.5	221
8.7	237
8.8	248
9.0	261
9.3	272
9.4	278
9.6	288
9.8	299
10.1	304
10.3	318
10.4	327
10.6	340
10.9	349

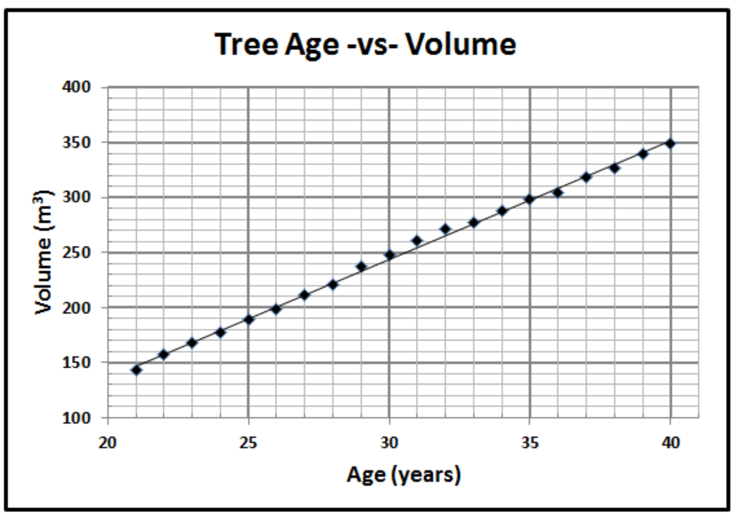
5. A tree's age and volume are related linearly as shown in the graph below.

Round slopes and y-intercepts to 3 decimal places

a) Find the equation for the trend line using the regression feature of your graphing calculator.

b) Use your regression equation to predict the volume of a 15 year old tree, accurate to 1 decimal place.

c) Use your regression equation to predict the age of a tree with a 500 m^3 volume, rounded to the nearest year.



Age	Volume (m^3)
21	143
22	157
23	168
24	178
25	189
26	199
27	212
28	221
29	237
30	248
31	261
32	272
33	278
34	288
35	299
36	304
37	318
38	327
39	340
40	349

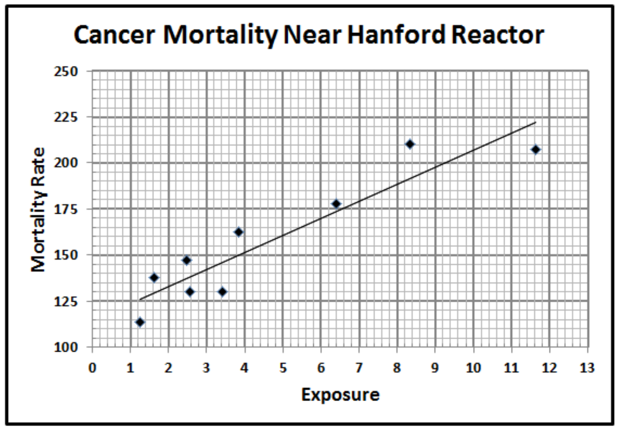
6. A study conducted in the early 1960's in 9 counties in northern Oregon and southern Washington related exposure to the Hanford nuclear plant and mortality rate. Exposure is an index number assigned to a county based on its population and proximity to the contaminated water. Mortality is the rate of deaths per 100,000 man-years.

Exposure	Mortality
2.5	147.1
2.6	130.1
3.4	129.9
1.3	113.5
1.6	137.5
3.8	162.3
11.6	207.5
6.4	177.9
8.3	210.3

a) Find the equation for the trend line using the regression feature of your graphing calculator. Round slope and y-intercept to 2 decimal places.

b) Use your regression equation to predict the mortality rate for an exposure index of 18, accurate to the nearest tenth.

c) Use your regression equation to predict the exposure index for a mortality rate of 300 (deaths per 100,000 man-years), accurate to the nearest tenth.



7. The winning Olympic discus throw is recorded through the 1900's rounded to the nearest foot.

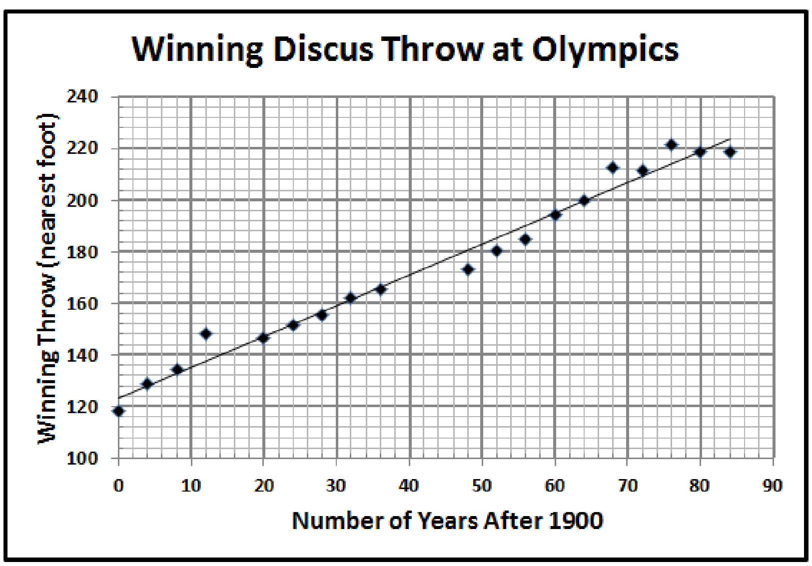
Round slopes and y-intercepts to 2 decimal places

a) Consider 1900 to be year 0 and find the equation for the trend line using the regression feature of your graphing calculator.

b) Use your regression equation to predict the winning throw at the 2020 Olympics, accurate to the tenth place.

c) Use your regression equation to predict the year the winning throw will reach 280 feet, accurate to the nearest year.

Year	Throw (ft)
1900	118
1904	129
1908	134
1912	148
1920	147
1924	151
1928	155
1932	162
1936	166
1948	173
1952	181
1956	185
1960	194
1964	200
1968	213
1972	211
1976	221
1980	219
1984	219



8. Consider the price list for rectangular tarps of different sizes.

   Round slopes and y-intercepts to 4 decimal places

   a) Find the area of each tarp in the chart (area of a rectangle is length times width).
   b) Find the equation for the trend line using the regression feature of your graphing calculator.
   c) Use your regression equation to calculate the price for the 20' x 24' tarp to the nearest dollar.
   d) Use your regression equation to calculate the price for the 24' x 30' tarp to the nearest dollar.

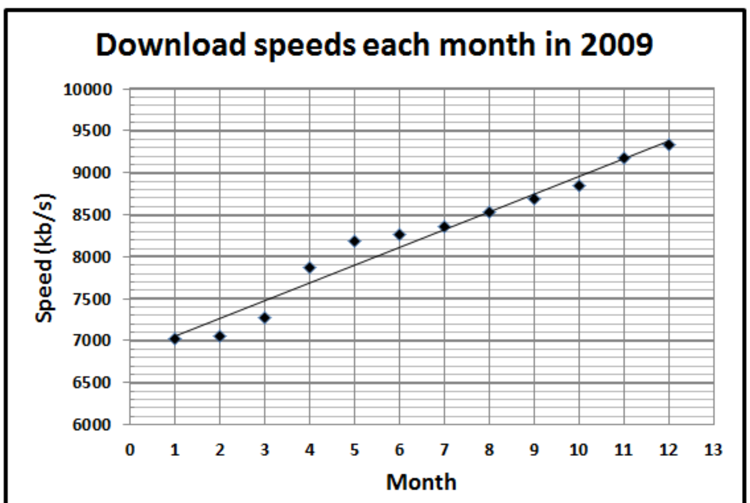
Dimensions	Area	Price
5' x 7'		$4
6' x 8'		$5
8' x 10'		$8
10' x 12'		$11
12' x 14'		$15
16' x 20'		$26
20' x 24'		
24' x 30'		

9. Download speeds for data are increasing with technological advances. The speeds are recorded in the table for each month in 2009.

   Round slopes and y-intercepts to 2 decimal places

   a) Find the equation for the trend line using the regression feature of your graphing calculator.
   b) Use your regression equation to predict the month the download speed will reach 11,000 Kb/s.
   c) Use your regression equation to predict the download speed at month 24 (the end of 2010), answer to the nearest Kb/s.

Month	Speed (Kb/s)
1	7028
2	7056
3	7278
4	7866
5	8188
6	8265
7	8355
8	8529
9	8694
10	8844
11	9183
12	9332



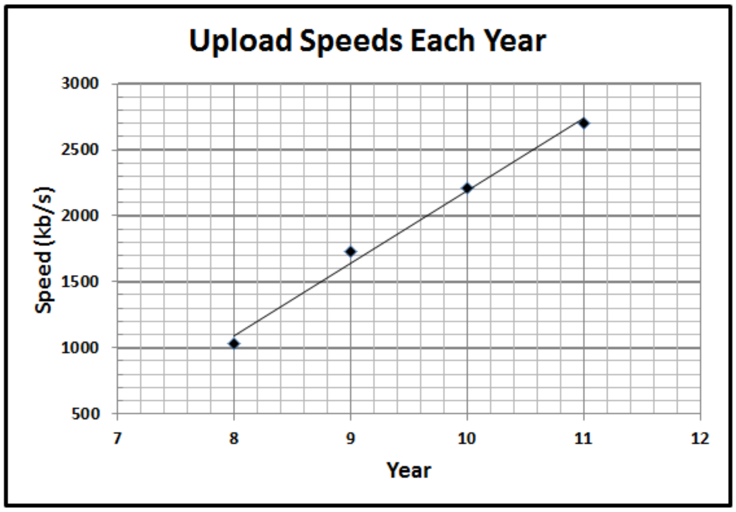
10. Upload speeds for data are increasing with technological advances. Consider the year 2000 to be year 0.

Year	Speed (Kb/s)
2008	1029
2009	1728
2010	2207
2011	2705

a) Find the equation for the trend line using the regression feature of your graphing calculator.

b) Use your regression equation to predict the upload speed in 2015 rounded the nearest Kb/s.

c) Use your regression equation to predict the year the upload speed will reach 6000 Kb/s.



11. Smoking rates for adults have been declining over the last couple of decades. Consider 1990 to be year 0.

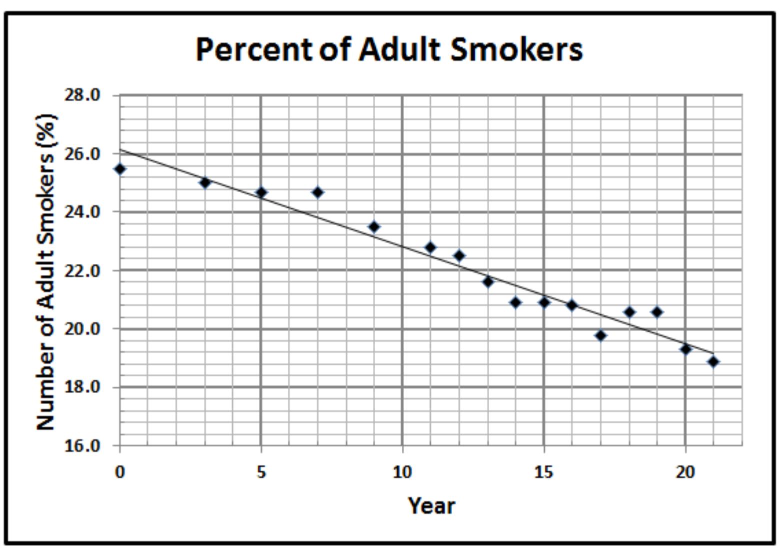
Year	Percent
1990	25.5
1993	25.0
1995	24.7
1997	24.7
1999	23.5
2001	22.8
2002	22.5
2003	21.6
2004	20.9
2005	20.9
2006	20.8
2007	19.8
2008	20.6
2009	20.6
2010	19.3
2011	18.9

Round slopes and y-intercepts to 3 decimal places

a) Find the equation for the trend line using the regression feature of your graphing calculator.

b) Use your regression equation to predict the percent of adult smokers in 2018 to the nearest tenth of a percent.

c) Use your regression equation to predict the year the smoking percentage will reach 10%.



12. Health costs are steadily rising in the United States. Consider 1990 as year 0.

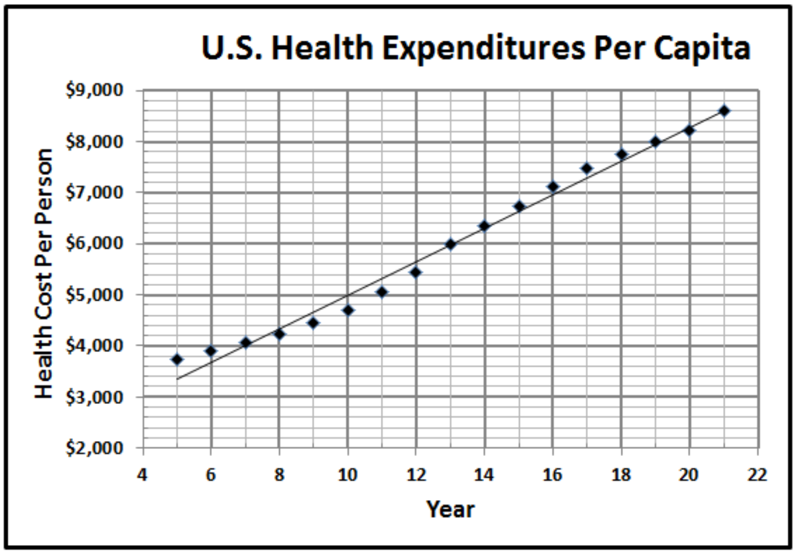
Round slopes and y-intercepts to 3 decimal places

a) Find the equation for the trend line using the regression feature of your graphing calculator.

b) Use your regression equation to predict the health expenditure in 2018, rounded to the nearest dollar.

c) Use your regression equation to predict the year the health expenditure will reach $10,000 per person.

Year	Expenditure
1995	$3,748
1996	$3,900
1997	$4,055
1998	$4,236
1999	$4,450
2000	$4,703
2001	$5,052
2002	$5,453
2003	$5,989
2004	$6,349
2005	$6,728
2006	$7,107
2007	$7,482
2008	$7,760
2009	$7,990
2010	$8,233
2011	$8,608



13. Smoking rates for students have been declining. Consider 1990 as year 0.

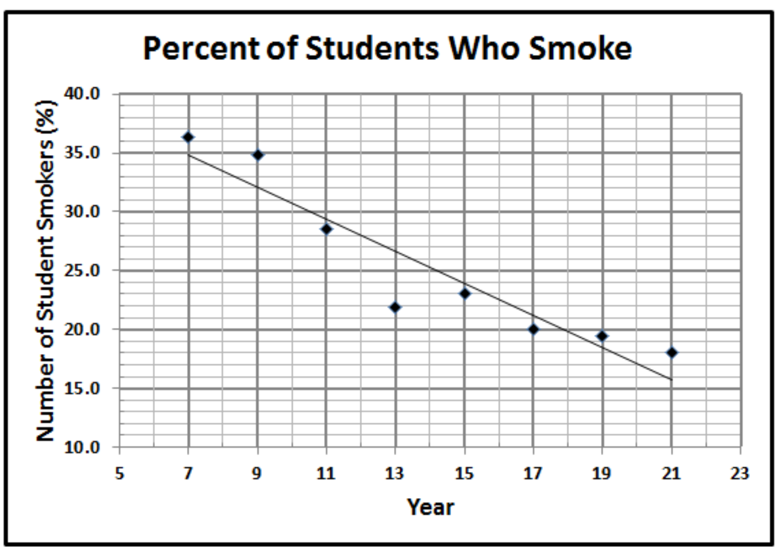
Round slopes and y-intercepts to 3 decimal places

a) Find the equation for the trend line using the regression feature of your graphing calculator.

b) Use your regression equation to predict the percent of student smokers in 2020, accurate to the hundredth place.

c) Use your regression equation to predict the year the smoking percentage will reach 14%.

Year	Percent
1997	36.4
1999	34.8
2001	28.5
2003	21.9
2005	23.0
2007	20.0
2009	19.5
2011	18.1



Chapter 2:

Quadratic Relationships

Aricebo Observatory in Puerto Rico boasts the largest satellite dish in the world at 1000 feet in diameter.

Most satellite dishes are rotated parabolas, which is the geometric shape for a <u>quadratic equation</u>. They possess the unique quality of collecting a signal and focusing it at a single point above the dish. It is likely that you have one of these on your house right now collecting your television signal.

There are two useful applications for the inimitable quadratic equation:

1. Collecting and sending radio, television, sound and light waves
2. Modeling data that reaches a maximum or minimum

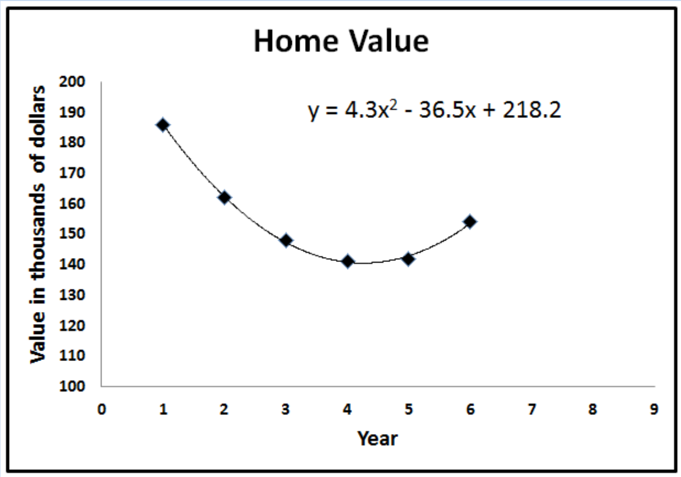
Let's reconsider the real estate application that has a minimum:

Example: Tracking Home Value

Consider the value of a family home for each of the first 6 years of ownership. Use the regression feature of your graphing calculator to find a quadratic equation to model the data. Use your equation to predict the value of the home in year 9.

Year	Home Value (in thousands $)
1	186
2	162
3	148
4	141
5	142
6	154

You may have fallen into a pattern assuming all data is linear since that was our focus in chapter one. The graph clearly shows the shape of this data would not be modeled very well by a line. This type of curve is called a **parabola** which is modeled by a **quadratic equation** of the form $y = ax^2 + bx + c$.



Notice the equation has an exponent of 2. I like to think of the exponent as relating to its shape in that it moves in 2 directions, (especially since more advanced math courses will show that an equation with an exponent of 3 will have a graph moving in 3 different directions).

Solution:

First, enter the data into your calculator behind the STAT button exactly as you did in chapter 1.

Second, this time choose "QuadReg" in the CALC menu which is short for quadratic regression since this data is modeled by a curve with 2 changes of direction. You should get the same equation that you see listed on the graph (rounded to the tenth place).

Third, now replace the x with 9 and consider the order of operations (PEMDAS).

$y = 4.3(9)^2 – 36.5(9) + 218.2$	substitute 9 for x
$y = 4.3(81) – 36.5(9) + 218.2$	exponents first
$y = 348.3 - 328.5 + 218.2$	multiplication next
$y = 238$	add/subtract from left to right

Final Answer: The house will be worth around $238,000 in year 9.

Note: this prediction assumes the market value continues on this pattern upward. It would not be at all unusual for the housing market to change pattern before then.

Section 2.1: The Shape of a Quadratic Equation

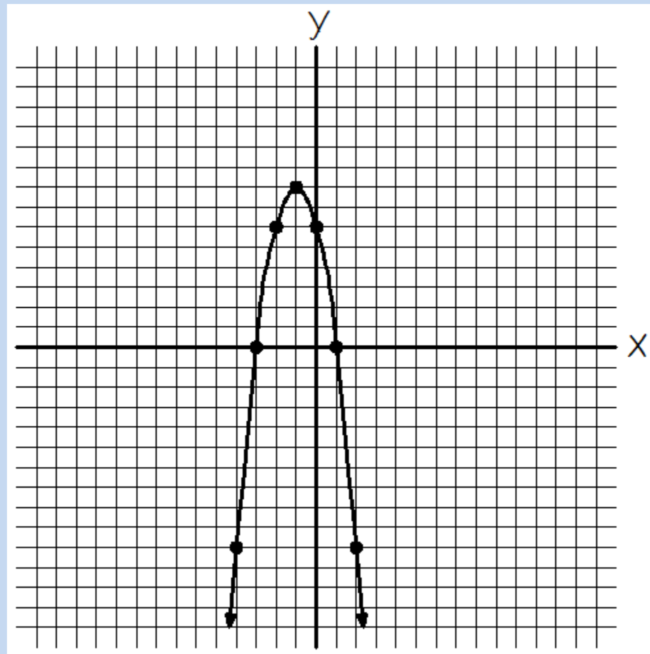
Let's examine a generic quadratic equation and find its graph:

Example 2.1.1: Graphing a Quadratic Equation

Find some ordered pairs then graph the equation: $y = -2x^2 - 4x + 6$

Solution:

You are free to pick any value for x. Remember to consider the order of operations when simplifying (PEMDAS).



$y = -2x^2 - 4x + 6$	let x = -4
$y = -2(-4)^2 - 4(-4) + 6$	replace x with -4
$y = -2(16) - 4(-4) + 6$	exponents first
$y = -32 + 16 + 6$	multiplication next
$y = -10$	add from left to right

x	y
-4	-10
-3	0
-2	6
-1	8
0	6
1	0
2	-10
3	-24

A partial list of ordered pairs is shown in the table.

We often choose whole numbers for x to make the calculation easier.

Can you replace x with 1.5 and get- 4.5 for y? Notice (1.5,-4.5) is also on the graph.

Can you replace x with $\frac{2}{3}$ and get $2\frac{4}{9}$ for y? Notice $\left(\frac{2}{3}, 2\frac{4}{9}\right)$ is also on the graph.

Note: Remember that we add the curve based on the pattern we see in the ordered pairs. The curve represents the infinite number of solutions to the equation.

Note: The graph has two x-intercepts: (-3,0) & (1,0)
The graph has one y-intercept: (0,6)
The graph has a **maximum**: (-1,8)
Some graphs will have **minimums**. This "turn around point" is formally referred to as the **vertex** whether maximum or minimum.

Note: Notice that the positive value for "a" in the home value example produced an upward facing parabola, while the graph of $y = -2x^2 - 4x + 6$, having a negative value for "a", produced a downward facing parabola.

Now a practical problem:

Example 2.1.2: Graphing a Quadratic Equation

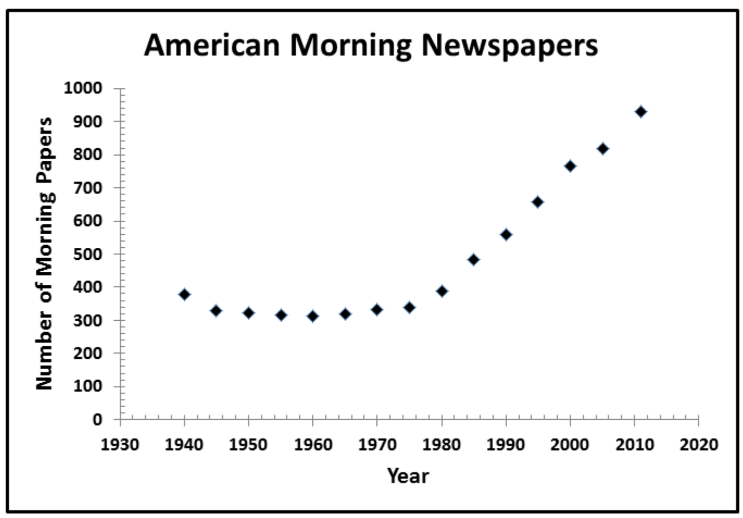
Consider the data regarding the number of morning newspaper companies in the United States since 1940.

1. Find and interpret the slope from 1940 to 1945.
2. Find and interpret the slope from 1950 to 1955.
3. Find and interpret the slope from 1990 to 2000.
4. Graph the data.
5. Estimate the vertex and explain its meaning.
6. Add a trend "line" and make a prediction for the number of companies in 2020.

Year	Morning Papers
1940	380
1945	330
1950	322
1955	316
1960	312
1965	320
1970	334
1975	339
1980	387
1985	482
1990	559
1995	656
2000	766
2005	817
2011	931

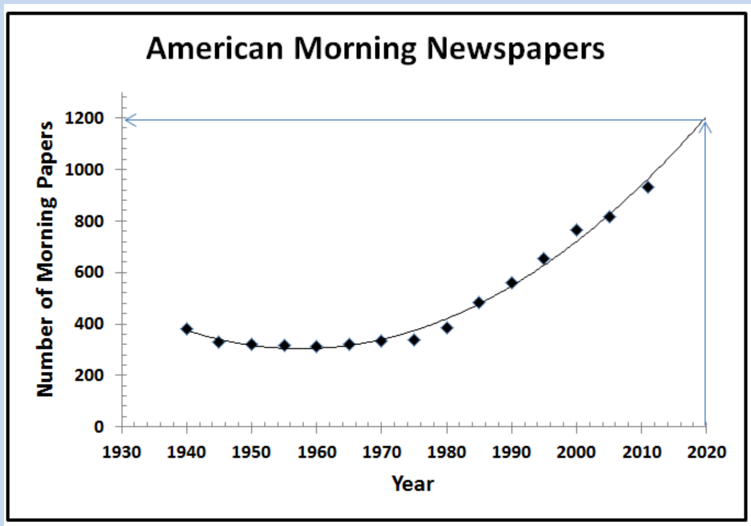
Solution:

1. $\frac{330-380}{1945-1940} = \frac{-50}{5} = \frac{-10}{1}$, meaning a **decrease** of 10 papers per year during this period of time
2. $\frac{316-322}{1955-1950} = \frac{-6}{5} = \frac{-1.2}{1}$, meaning a **decrease** of about 1 paper per year during this period of time
3. $\frac{766-559}{2000-1990} = \frac{207}{10} = \frac{20.7}{1}$, meaning an **increase** of about 21 papers per year during this period of time
4. See graph



5. The vertex appears to be about 1960 when there were 312 morning newspaper companies. This is the least number of companies in American history.

6. The trend would indicate that there will be about 1200 morning newspaper companies in 2020. The phrase "trend line" will be used in every section even though it may not actually be a line.



Section 2.1: Problem Set

1. American newspaper circulation enjoyed continuous growth until the 1970's where it roughly leveled off then began a steady decline in the 1990's, presumably due to the internet.

Year	Circulation
1940	41132
1945	48384
1950	53829
1955	56147
1960	58882
1965	60358
1970	62108
1975	60655
1980	62202
1985	62766
1990	62328
1995	58193
2000	55773
2005	53345
2011	44421

   a) Make a graph of the data (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
   b) Find the slope between 1940 and 1945. Explain the meaning of the slope in context.
   c) Find the slope between 1960 and 1965. Explain the meaning of the slope in context.
   d) Find the slope between 1995 and 2000. Explain the meaning of the slope in context.
   e) Add a trend line to the graph.
   f) Estimate the vertex and explain its meaning in context.
   g) Estimate both x-intercepts and explain their meaning in context.

2. The apparent temperature or heat index is a measure of how hot it feels despite what the thermometer says. The more moisture that is in the air the hotter it will feel. The chart shows the heat index for a temperature of 90^0 as a function of humidity.

Humidity (%)	Heat Index (F^0)
40	91
45	93
50	95
55	97
60	100
65	103
70	105
75	109
80	113
85	117
90	122
95	127
100	132

   a) Make a graph of the data (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
   b) Find the slope between 40% and 50% humidity. Explain the meaning of the slope in context.
   c) Find the slope between 90% and 100% humidity. Explain the meaning of the slope in context.
   d) Add a trend line to the graph.

3. The number of inmates incarcerated in United States prisons is shown in the table.

Year	Prisoners (in thousands)
2005	1448
2006	1493
2007	1518
2008	1522
2009	1525
2010	1521
2011	1504

   a) Make a graph of the data (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
   b) Find the slope between 2005 and 2007. Explain the meaning of the slope in context.
   c) Find the slope between 2009 and 2011. Explain the meaning of the slope in context.
   d) Add a trend line to the graph.
   e) Estimate the vertex and explain its meaning in context.

4. The percent of high school students who smoke is shown in the table.

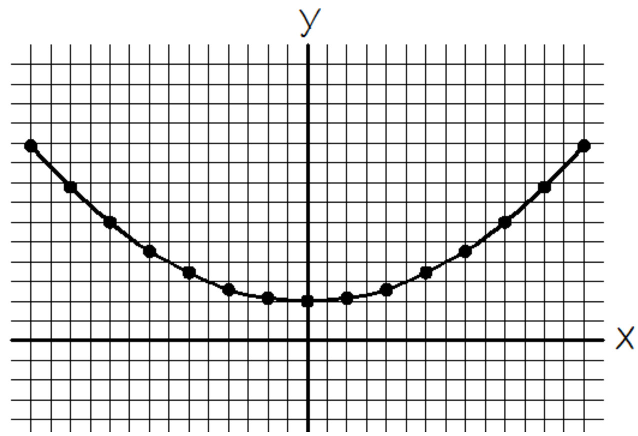
Year	Percent
1993	30.5
1995	34.8
1997	36.4
1999	34.8
2001	28.5
2003	21.9

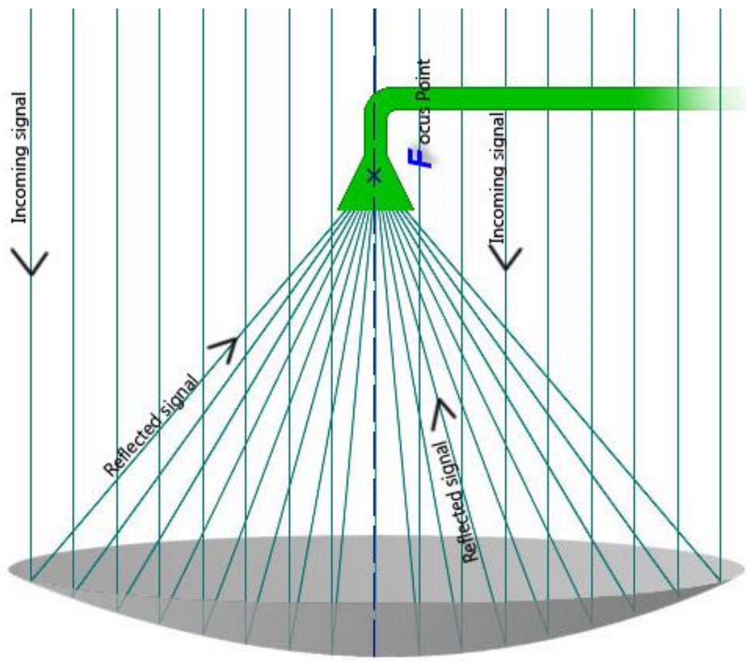
   a) Make a graph of the data large enough to include 2010 (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
   b) Find the slope between 1993 and 1997, accurate to 1 decimal place. Explain the meaning of the slope in context.
   c) Find the slope between 1999 and 2003, accurate to 1 decimal place. Explain the meaning of the slope in context.
   d) Add a trend line to the graph.
   e) Estimate the vertex and explain its meaning in context.
   f) Estimate the right side x-intercept and explain its meaning in context.

5. A satellite dish is a precise mathematical shape (a parabola) that has the property of focusing a reflected signal at a single point. The scale drawing of a cross section of a 28 foot wide parabolic satellite dish is shown on the graph where each square is 1ft x 1ft. The dish is designed using the equation $y = .04x^2 + 2$. The ordered pair solutions of the equation become the dimensions necessary to build this amazing device.

   a) Treat the x-axis as the ground and use the equation to find the height of the dish above the ground at the 2 foot intervals shown on the graph.

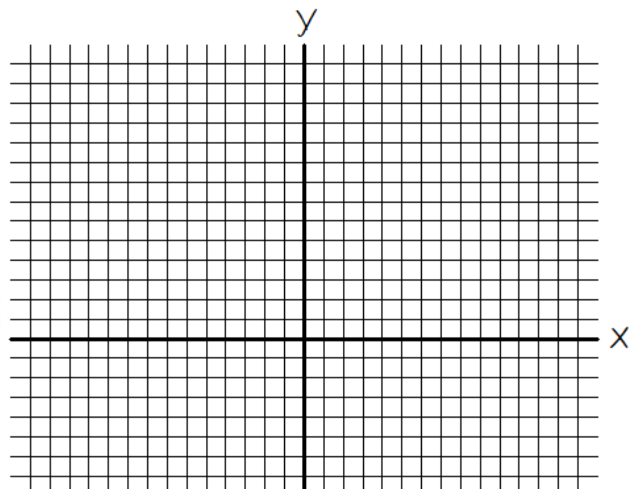
   b) Recall that the quadratic equation $y = ax^2 + bx + c$ defines the geometric parabola. It turns out that the focus point is located a distance $\frac{1}{4a}$ above the bottom of the dish. Find this distance for this satellite dish.





6. Make the scale drawing of a cross section of a 24 foot wide parabolic satellite dish where each square is 1ft x 1ft. The dish is designed using the equation $y = .05x^2 + 3$.

   a) List at least 13 ordered pairs.
   b) Include the focus point on your graph (refer to the previous question for the formula).



Section 2.2: Finding Quadratic Equations

Algebra is all about expressing relationships as ordered pairs, graphs and equations. In section 2.1 we graphed ordered pairs that we were given or found for ourselves from an equation. In this section we learn how to find the equation from ordered pairs or a graph.

In section 1.2 we learned that there is a direct connection between the graph of a line and its equation. Miraculously, there is a similar relationship for parabolas. If we graph a quadratic equation in a slightly different form perhaps you will discover it for yourself.

Example 2.2.1: Graphing a Quadratic Equation

Find some ordered pairs then graph the equation: $y = 2(x - 3)^2 - 8$

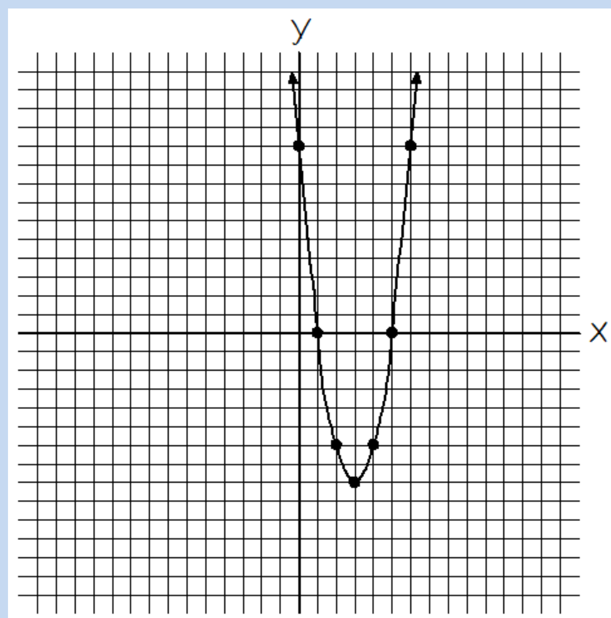
Solution:

You are free to pick any value for x. Remember to consider the order of operations when simplifying (PEMDAS).

$y = 2(x - 3)^2 - 8$	let x = -1
$y = 2(-1 - 3)^2 - 8$	replace x with -1
$y = 2(-4)^2 - 8$	parenthesis first
$y = 2(16) - 8$	exponents next
$y = 32 - 8$	multiplication next
$y = 24$	add/subtract

A partial list of ordered pairs is shown in the table.

x	y
-1	24
0	10
1	0
2	-6
3	-8
4	-6
5	0
6	10



Note: Remember that we add the curve based on the pattern we see after graphing the ordered pairs. The curve represents the infinite number of solutions to the equation even though we only found 8.

Compare the vertex of the parabola (3, -8) with its equation $y = 2(x - 3)^2 - 8$**.**
It turns out that if the equation is in its **standard form**: $\mathbf{y = a(x - h)^2 + k}$; that (h, k) will be the vertex. All that is necessary to find "a" is to substitute some other representative point for x and y and solve for "a".

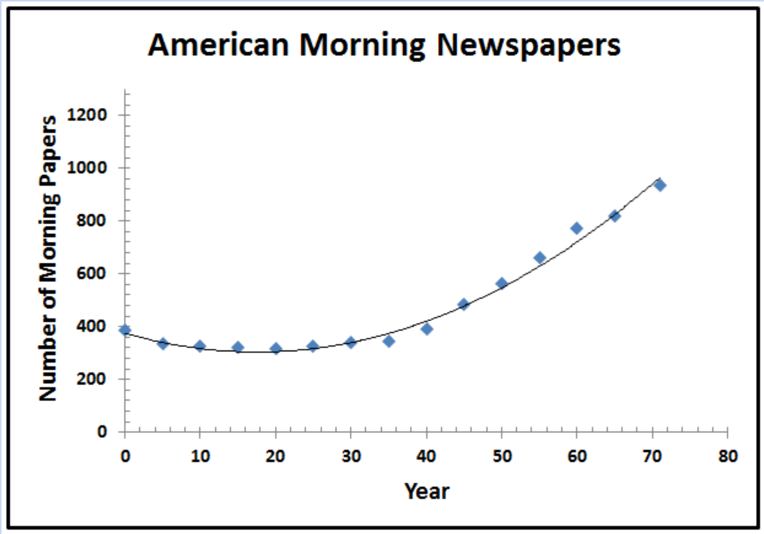
Now a practical problem:

Example 2.2.2: Finding a Quadratic Equation

Consider the data regarding the number of morning newspaper companies in the United States since 1940. Find an equation by hand to model the data. Then use the equation to predict the number of morning papers there will be in 2025. For simplicity we can consider 1940 to be year 0.

Year	Morning Papers
1940	380
1945	330
1950	322
1955	316
1960	312
1965	320
1970	334
1975	339
1980	387
1985	482
1990	559
1995	656
2000	766
2005	817
2011	931

Solution:



Having graphed the data, you can see the point (20,312) is a reasonable estimate for the vertex, substituting into the equation $y = a(x - h)^2 + k$ we have: $y = a(x - 20)^2 + 312$.

The graph passes through the point (65,817). Substituting we have:

$817 = a(65 - 20)^2 + 312$	substitute 65 for x and 817 for y
$817 = 2025a + 312$	simplifying $(65 - 20)^2$
$505 = 2025a$	subtract 312 from both sides
$a \approx .249$	divide both sides by 2025

The equation $y = .249(x - 20)^2 + 312$ is a fairly good model for the data.

$y = .249(x - 20)^2 + 312$	year 2025 would mean x = 85
$y = .249(85 - 20)^2 + 312$	replace x with 85
$y = .249(65)^2 + 312$	parenthesis first
$y = .249(4225) + 312$	exponents next
$y = 1052.025 + 312$	multiplication next
$y = 1364.025$	add

Final Answer: If the pattern of growth continues in a similar fashion, there will be approximately 1364 morning newspaper companies in 2025.

Finding an equation by hand using the standard form of a parabola is a confidence builder, but as we saw in the last chapter using regression is much easier and more accurate.

Let's try this again with regression:

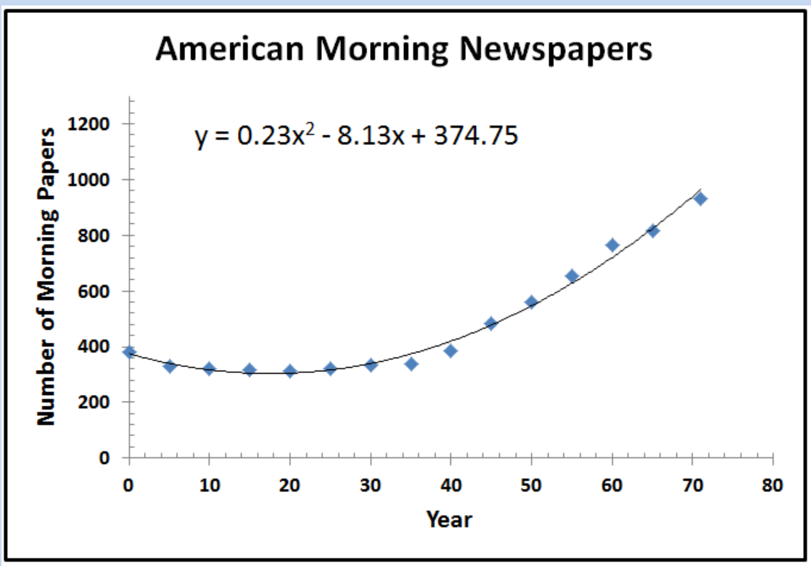
Example 2.2.3: Finding a Quadratic Equation

Consider the data regarding the number of morning newspaper companies in the United States since 1940. Use the regression feature of your graphing calculator to find an equation to model the data. Then use the equation to predict the number of morning papers there will be in 2025.

Year	Morning Papers
1940	380
1945	330
1950	322
1955	316
1960	312
1965	320
1970	334
1975	339
1980	387
1985	482
1990	559
1995	656
2000	766
2005	817
2011	931

Solution:

For simplicity we can consider 1940 to be year 0. Enter the 15 ordered pairs into the STAT feature of your calculator and choose (QuadReg). See if you agree with the equation listed on the graph (rounded to 2 decimal places).



Note: There is nothing wrong with leaving the years as given, but you get unnecessarily large numbers in the equation.

$y = .23x^2 - 8.13x + 374.75$	year 2025 would mean $x = 85$
$y = .23(85)^2 - 8.13(85) + 374.75$	replace x with 85
$y = .23(7225) - 8.13(85) + 374.75$	exponents first
$y = 1661.75 - 691.05 + 374.75$	multiplication next
$y = 1345.45$	add/subtract from left to right

Final Answer: If the pattern of growth continues in a similar fashion, there will be approximately 1345 morning newspaper companies in 2025.

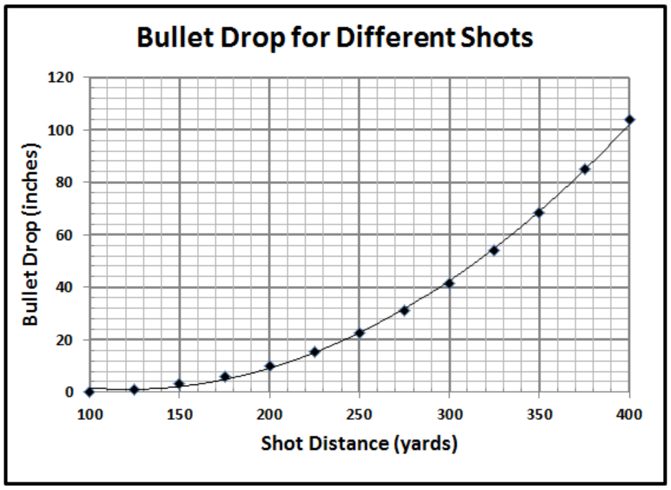
Note: The equation from example 2.2.2 can be simplified to look very much like the equation we found here in example 2.2.3. Ask your instructor to show you.

Section 2.2: Problem Set

1. The chart shows the inches of drop in a bullet for different length shots for a particular type of Hornady bullet design.

Hornady XTP 20 gr. 17 HMR MV=2375		
Click	Size	0.125
Yard	Drop	Click
100	0.0	0
125	1.2	8
150	3.2	17
175	6.1	28
200	10.2	41
225	15.3	55
250	22.6	73
275	31.2	91
300	41.7	112
325	54.1	134
350	68.3	157
375	84.9	182
400	103.7	208

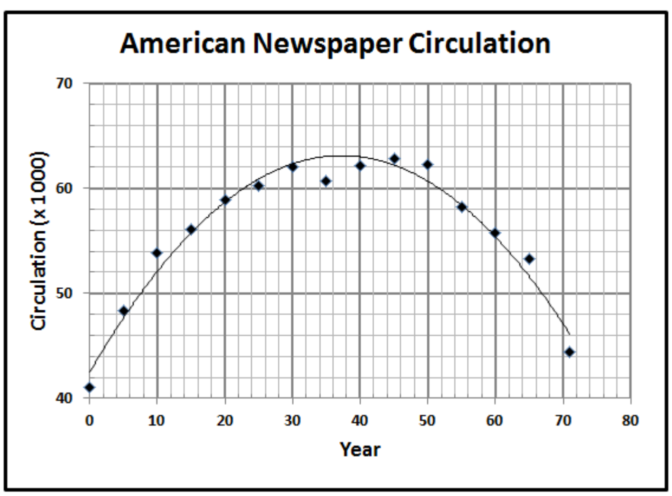
   a) Use regression to find a quadratic equation to model the data, considering drop as a function of yardage. Round the numbers in your equation to 4 decimal places.
   b) Use your equation to predict the drop for a 215 yard shot, accurate to 1 decimal place.
   c) Use your equation to predict the drop for a 500 yard shot, accurate to 1 decimal place.



2. American newspaper circulation enjoyed continuous growth until the 1970's where it roughly leveled off then began a steady decline in the 1990's, presumably due to the internet.

Year	Circulation (in thousands)
1940	41.1
1945	48.4
1950	53.8
1955	56.1
1960	58.9
1965	60.3
1970	62.1
1975	60.7
1980	62.2
1985	62.8
1990	62.3
1995	58.2
2000	55.8
2005	53.3
2011	44.4

   a) Consider 1940 to be year 0 and use regression to find a quadratic equation to model the data. Round the numbers in your equation to 3 decimal places.
   b) Use your equation to predict the circulation in 2020.

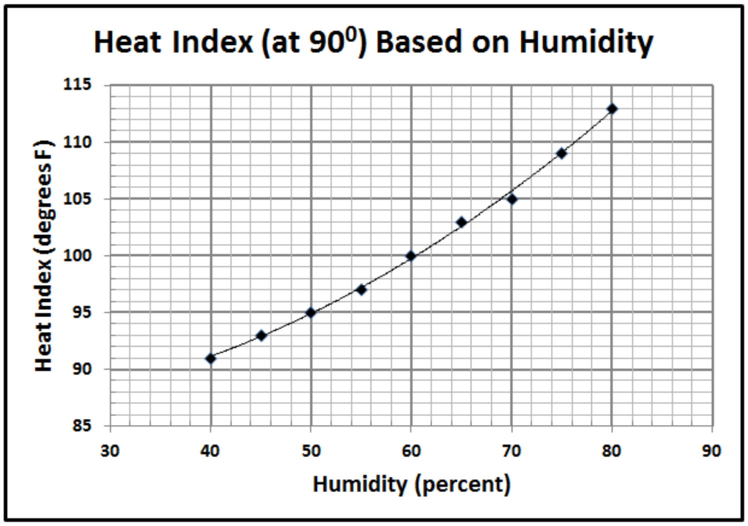


3. The apparent temperature or heat index is a measure of how hot it feels despite what the thermometer says. The more moisture that is in the air the hotter it will feel. The chart shows the heat index for a temperature of 90^0 as a function of humidity.

Humidity (%)	Heat Index (F^0)
40	91
45	93
50	95
55	97
60	100
65	103
70	105
75	109
80	113

a) Use regression to find a quadratic equation to model the data. Round the numbers in your equation to 4 decimal places.

b) Use your equation to predict the heat index at 85% humidity.



4. The number of inmates incarcerated in United States prisons is shown in the table.

Year	Prisoners (in thousands)
2005	1448
2006	1493
2007	1518
2008	1522
2009	1525
2010	1521
2011	1504

a) Consider 2000 to be year 0 and use regression to find a quadratic equation to model the data. Round the numbers in your equation to 2 decimal places.

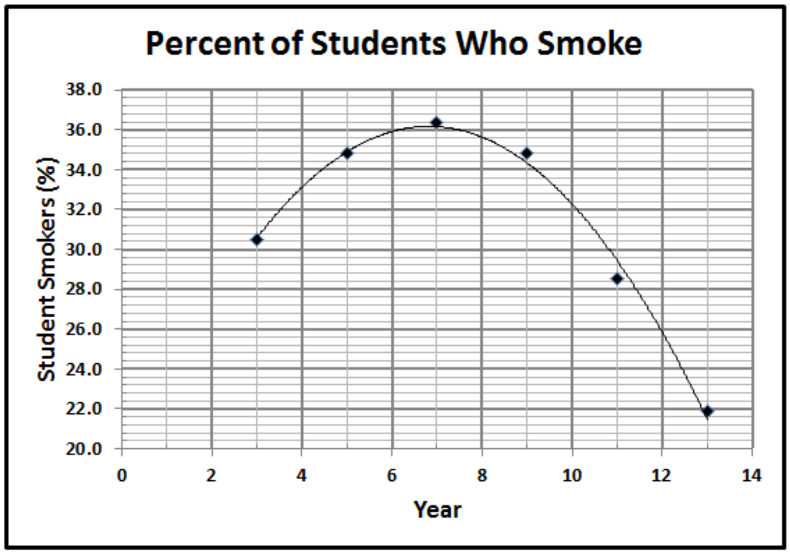
b) Use your equation to predict the number of prisoners in 2020.

5. The percent of high school students who smoke is shown in the table.

Year	Percent
1993	30.5
1995	34.8
1997	36.4
1999	34.8
2001	28.5
2003	21.9

a) Consider 1990 to be year 0 and use regression to find a quadratic equation to model the data. Round the numbers in your equation to 2 decimal places.

b) Use your equation to estimate the percent of students who smoked in 1988.

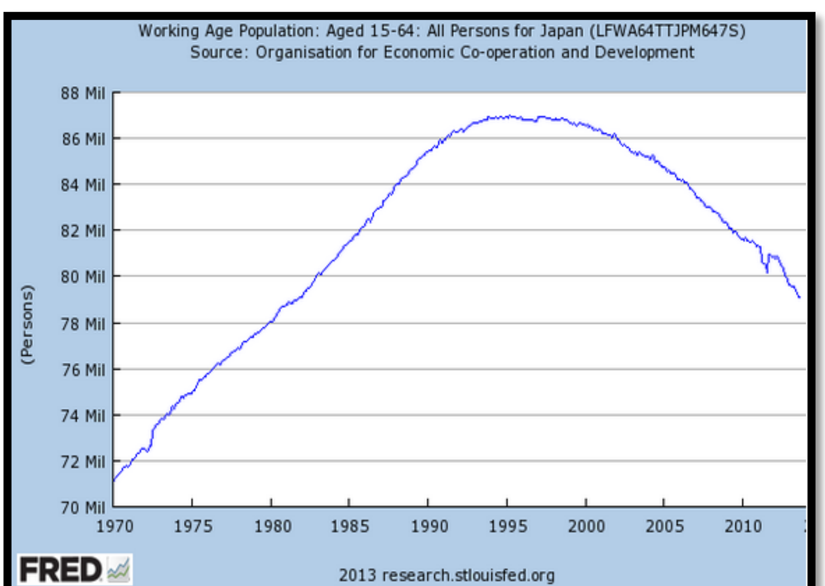


6. With zero immigration and declining birth rates, Japan's working-age population has declined sharply since 1995.

Year after 1970	Million Persons
0	71
5	75
10	78
15	82
20	85
25	87
30	87
35	85
40	82

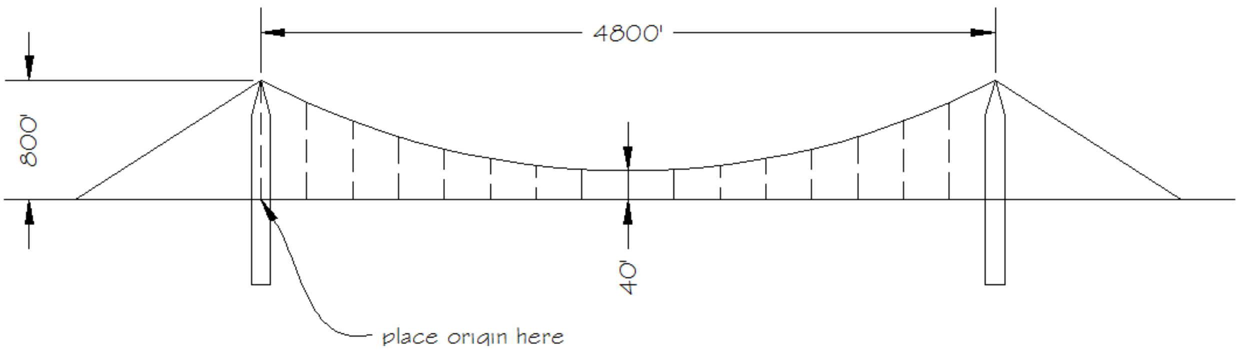
a) Consider 1970 to be year 0 and use regression to find a quadratic equation to model the data. Round the numbers in your equation to 2 decimal places.

b) Use your equation to estimate the population of working age people in 2020.

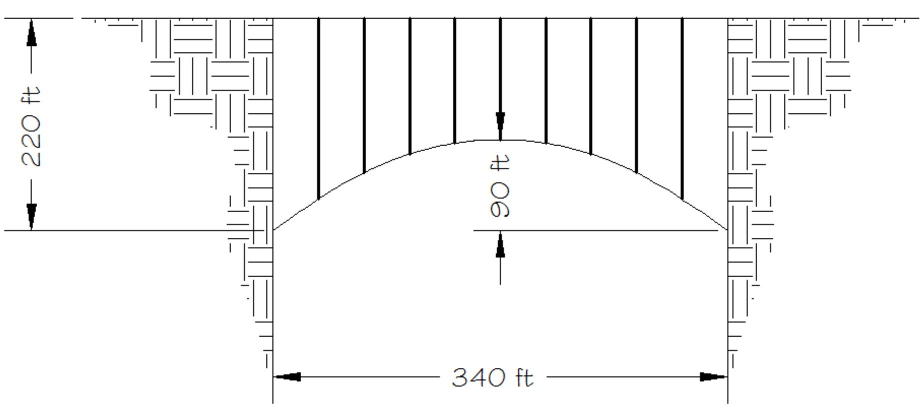


7. The golden gate is a suspension bridge where the road is supported by <u>equally spaced</u> cables hanging from a large parabolic cable, which is itself supported by 2 large towers.

   a) Use the dimensions below (measured in feet) to find 3 ordered pairs that define the parabola, then regression to find the equation for the parabola, accurate to 6 decimal places.
   b) Use the equation to find the lengths of the 7 hanging cables between the left tower and the 40 foot center cable. Round to 2 decimal places.

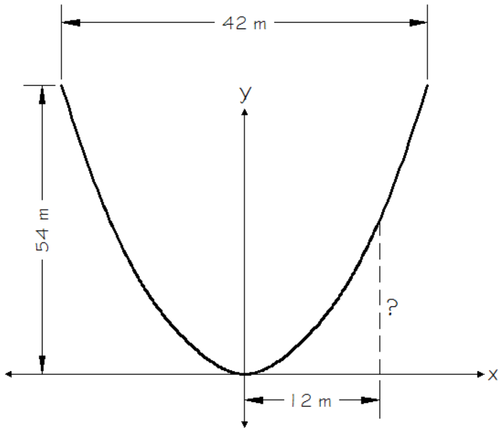


8. A bridge is to be supported by 9 evenly spaced vertical columns resting on a parabolic concrete arch. Choose an origin, find an equation the parabola based on the dimensions (round decimals to 7 places); and find the lengths of the 9 vertical columns (round decimals to 1 place).



9. A parabolic satellite dish is to be built 42 meters wide and 54 meters deep.

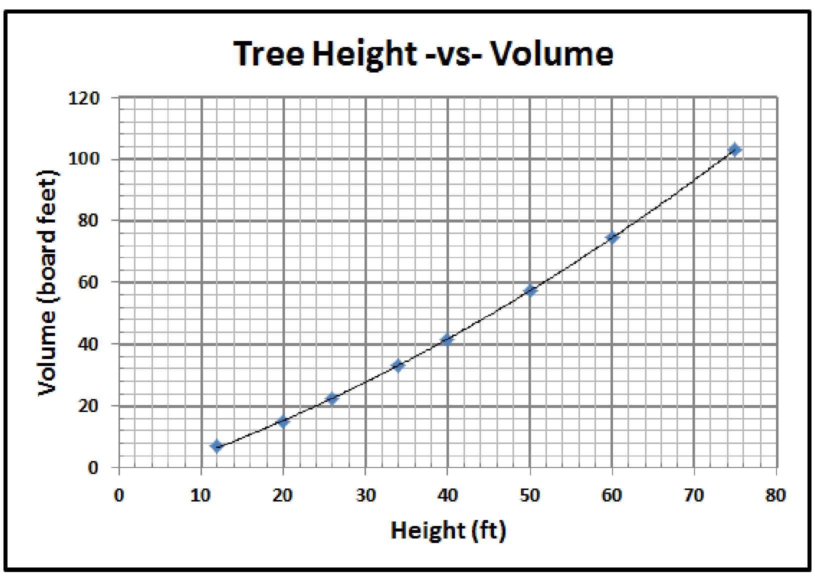
   a) Find the length of a support that is 12 meters away from the vertex, accurate to the hundredth place.
   b) Find the height of the focus above the vertex, accurate to the hundredth place. Recall that the focus point is located a distance $\frac{1}{4a}$ above the bottom of the dish.



10. The volume of a tree can be estimated knowing its height. Lumber is commonly measured in board feet (one board foot is 12" x 12" x 1").

   a) Use regression to find a quadratic equation to model the data. Round the numbers in your equation to 3 decimal places.
   b) Use your equation to predict the number of board feet a 90 foot tree would be expected to have. Round to the nearest board foot.

Height (ft)	Volume (BF)
12	6.8
20	15.0
26	22.3
34	32.9
40	41.8
50	57.6
60	74.8
75	102.9



Section 2.3: Using Quadratic Equations

Algebra is all about expressing relationships as ordered pairs, graphs and equations. In section 2.2 we used regression to find an equation from ordered pairs then used the equation to find a new ordered pair that represented a prediction for the future. In every case our prediction involved finding the y-value when x was known, which only required us to know the order of operations. In this section we tackle the surprisingly complicated problem of finding the x-value when the y is known.

Months are required in typical algebra courses to provide the background necessary to completely understand and even create the tool that will allow us to navigate this challenge. There are no shortcuts, however, so we will only look into it enough to get the basic idea.

Consider the equation: $y = x^2 - 6x + 9$ and suppose we wish to know the value(s) of x that will let y = 16.

We could just guess until we get lucky.

$(5)^2 - 6(5) + 9 = 4$	guess 5 … no!
$(2)^2 - 6(2) + 9 = 1$	guess 2 … no!
$(8)^2 - 6(8) + 9 = 25$	guess 8 … no!
$(7)^2 - 6(7) + 9 = 16$	guess 7 … yes!

Guessing is a waste of time and will not be very accurate when x is a decimal, and mathematicians, in an admittedly strange reversal of fortunes, consider guessing to be nerdy.

We could try to solve it using normal algebra techniques:

$16 = x^2 - 6x + 9$	replace y with 16
$7 = x^2 - 6x$	subtract 9 from both sides

Now we hit a wall. There are two x's in the equation that we cannot collect together. Solving for x requires that we isolate x, which is impossible to do in this way.

Mathematicians devised a brilliant solution, by noticing that you can factor $x^2 - 6x + 9$ as a perfect square, just like you can factor 36 (6•6 or $6^2 = 36$).

$(x - 3)(x - 3) = x^2 - 3x - 3x + 9$	distributive law is important here
$x^2 - 6x + 9$	collecting like terms
$x^2 - 6x + 9 = 16$	our problem
$(x - 3)(x - 3) = 16$	factor $x^2 - 6x + 9$
$(x - 3)^2 = 16$	both factors are the same so we can use exponents to rewrite it and have only one x
$x - 3 = \pm 4$	take the square root of both sides
$x = 3 + 4$ or $x = 3 - 4$	add 3 to both sides
$x = 7$ or -1	one of which we guessed earlier

This problem was <u>very</u> canned in that it was easy to take the square root of 16, and $x^2 - 6x + 9$ factored into a perfect square. If the 9 is changed to an 8, for instance, $x^2 - 6x + 8$ only factors to $(x - 4)(x - 2)$, which is not helpful.

Months are spent getting good at factoring and learning how to modify expressions so they will factor into perfect squares (a process known as *completing the square*). If you get good enough that you can factor and complete the square with letters instead of numbers, you arrive at the proverbial mountain top; the most holy **quadratic formula**, one of the most beautiful results ever created by mathematicians.

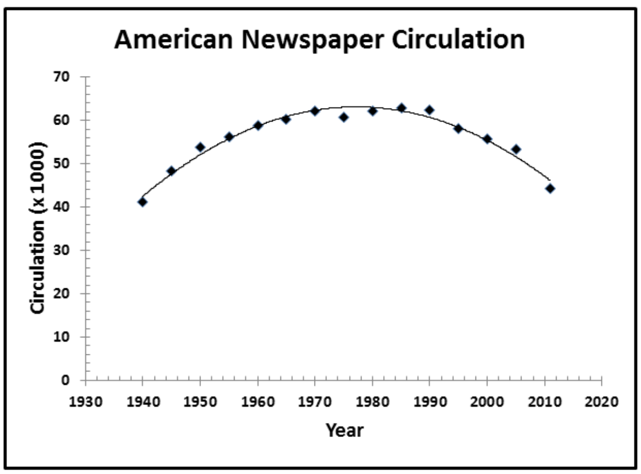
The Quadratic Formula:

When an equation is in the form $ax^2 + bx + c = 0$, where a, b & c are any numbers, the formula to solve for x is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Notice the equation we start with must equal 0 for this to work.

The Quadratic Formulas is one of the true gems of algebra. We no longer have to factor and complete the square, since we have done it once and for all with letters instead of numbers.



Quadratic equations typically have 2 solutions; one using the + sign and one using the - sign in the quadratic formula. Notice in the trend line that there were two years when American newspaper circulation was at 50 (thousand).

Let's return to the equation $x^2 - 6x + 9 = 16$, where we already learned that x = 7 & -1, and use the quadratic formula to arrive at the same result:

<u>Example 2.3.1: Using the Quadratic Formula</u>

Solve: $x^2 - 6x + 9 = 16$

Solution:

$x^2 - 6x - 7 = 0$	subtract 16 from both sides
$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-7)}}{2(1)}$	substitute a = 1, b = -6, c = -7
$x = \frac{-(-6) \pm \sqrt{64}}{2}$	simplifying inside the square root
$x = \frac{6 \pm 8}{2}$	simplifying $\sqrt{64}$ and -(-6)
$x = \frac{14}{2}$ and $\frac{-2}{2}$	evaluating 6 + 8 and 6 - 8
x = 7 and x = -1	dividing

Final Answer: x = 7 & -1

Let's try an example that doesn't come out as clean:

Example 2.3.2: Using the Quadratic Formula

Solve: $3x^2 + 12x - 3 = 27$

Solution:

$3x^2 + 12x - 30 = 0$	subtract 27 from both sides
$x = \frac{-12 \pm \sqrt{(12)^2 - 4(3)(-30)}}{2(3)}$	substitute a = 3, b = 12, c = -30
$x = \frac{-12 \pm \sqrt{504}}{6}$	simplifying inside the square root
$x \approx \frac{-12 \pm 22.45}{6}$	evaluating $\sqrt{504}$ with a calculator
$x \approx \frac{10.45}{6}$ and $\frac{-34.45}{6}$	evaluating -12 + 22.45 and -12 - 22.45
$x \approx 1.74$ & -5.74	dividing

Final Answer: $x \approx 1.74$ & -5.74

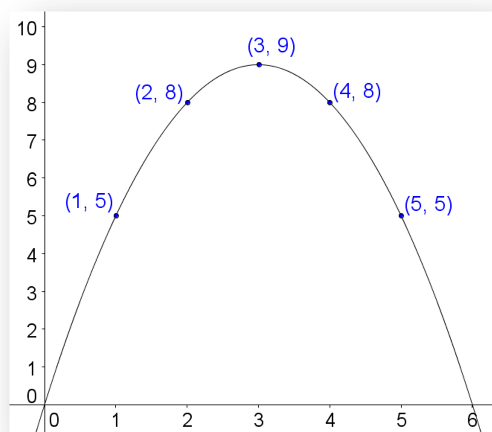
Note: Square roots often result in irrational numbers (non-terminating decimals without pattern) that must be rounded. It is a good practice to use the approximately equal sign (≈) to alert others, and remind yourself, that your answer is an approximation.

If you plug 1.74 into the original problem $3(1.74)^2 + 12(1.74) - 3$, you get 26.9628, which is *close* to 27.

The vertex is a very interesting point on a parabola, as it represents the maximum or minimum. The vertex is unique in that it is the only point on the graph that has a y value for which there is only one x value.

It can be found algebraically by reconsidering the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The plus and minus sign in the formula generate two answers.

The formula will produce one value if $b^2 - 4ac = 0$.



The vertex formula takes advantage of this observation:

The Vertex Formula:

The x-coordinate for the vertex in a quadratic equation of the form $y = ax^2 + bx + c$, is found with the formula:

$$\mathbf{x = \frac{-b}{2a}}$$

Substitute the value for x back into the equation to find the y-coordinate.

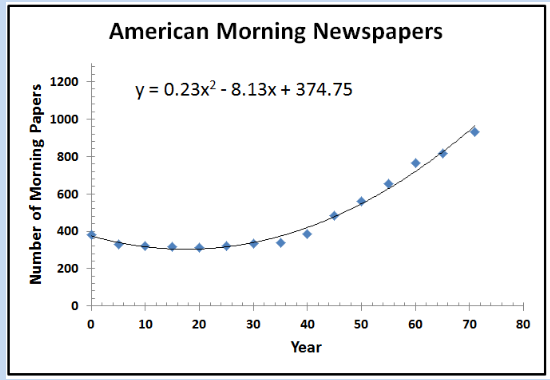
Now a practical problem:

Example 2.3.3: Using the Quadratic Formula

Consider the data regarding the number of morning newspaper companies in the United States since 1940, where 1940 is considered to be year 0.

a) Find the year(s) when the number of morning papers is 1200.

b) Find the vertex and interpret its meaning in context.



Solution a):

$y = .23x^2 - 8.13x + 374.75$

$1200 = .23x^2 - 8.13x + 374.75$ — replace y with 1200

$.23x^2 - 8.13x - 825.25 = 0$ — subtract 1200 from both sides

$x = \frac{-(-8.13)\pm\sqrt{(-8.13)^2-4(.23)(-825.25)}}{2(.23)}$ — substitute a = .23, b = -8.13, c = -825.25

$x \approx \frac{8.13\pm\sqrt{825.33}}{.46}$ — simplifying inside the square root

$x \approx \frac{8.13\pm 28.73}{.46}$ — simplifying $\sqrt{825.33}$ with a calculator

$x \approx \frac{36.86}{.46}$ and $\frac{-20.6}{.46}$ — evaluating 8.13 + 28.73 and 8.13 - 28.73

$x \approx 80.13$ & -44.78 — dividing

Final Answer: 80.1 years **after** 1940 (notice this looks reasonable on the graph) = 2020.1
44.8 years **before** 1940 = 1895.2

Note: Again this assumes that the pattern continues in the past and future based on the data. In this case it would be realistic to accept that there will be 1200 newspapers in 2020 but perhaps unrealistic to think there were that many in 1895.

Solution b):

$x = \frac{-b}{2a}$ — vertex formula

$x = \frac{-(-8.13)}{2(.23)}$ — a = .23, b = -8.13

$x \approx 17.7$ — simplifying

$y \approx .23(17.7)^2 - 8.13(17.7) + 374.75$ — substituting the value for x back into the original equation

$y \approx 303$ — simplifying

Final Answer: The vertex is at the point (17.7,303) … which is verified by the graph. Meaning that late in the year 1957 the country reached a minimum of about 303 paper companies.

Section 2.3: Problem Set

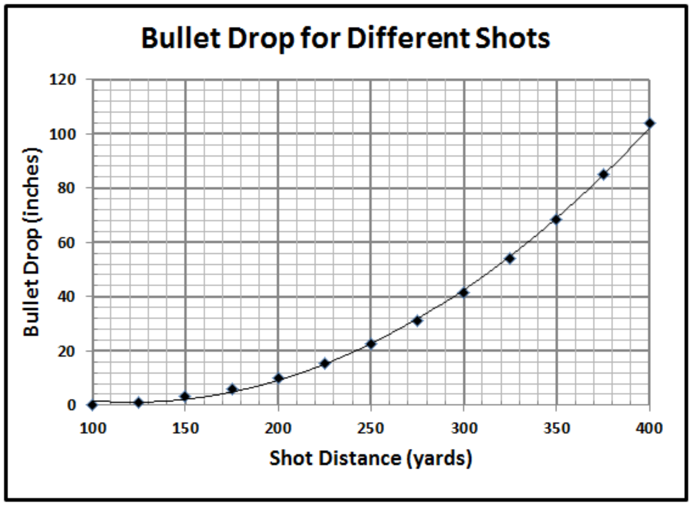
1. The chart shows the inches of drop in a bullet for different length shots for a particular type of Hornady bullet design.

Hornady XTP 20 gr. 17 HMR MV=2375		
Click	Size	0.125
Yard	Drop	Click
100	0.0	0
125	1.2	8
150	3.2	17
175	6.1	28
200	10.2	41
225	15.3	55
250	22.6	73
275	31.2	91
300	41.7	112
325	54.1	134
350	68.3	157
375	84.9	182
400	103.7	208

   a) Use regression to find a quadratic equation to model the data, considering drop as a function of yardage. Round the numbers in your equation to 4 decimal places.

   b) Use your equation to predict the drop for a 170 yard shot, accurate to 1 decimal place.

   c) Use your equation to estimate the yardage of a shot that dropped 140 inches, accurate to 1 decimal place.

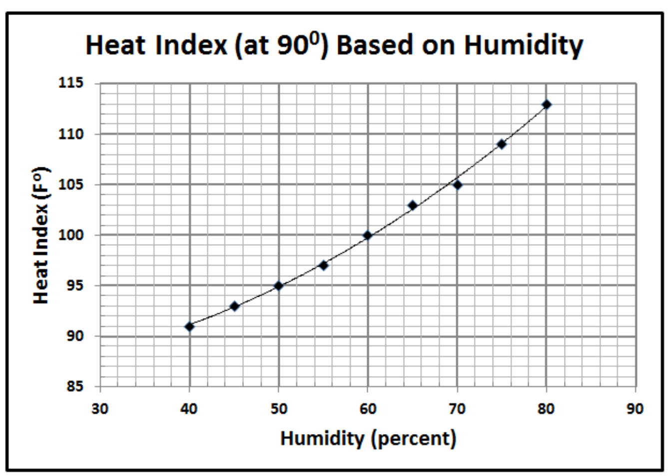


2. The apparent temperature or heat index is a measure of how hot it feels despite what the thermometer says. The more moisture that is in the air the hotter it will feel. The chart shows the heat index for a temperature of 90^0 as a function of humidity.

Humidity (%)	Heat Index (F^0)
40	91
45	93
50	95
55	97
60	100
65	103
70	105
75	109
80	113

   a) Use regression to find a quadratic equation to model the data. Round the numbers in your equation to 3 decimal places.

   b) Use your equation to predict the humidity required for a heat index of 125° F, accurate to 1 decimal place.



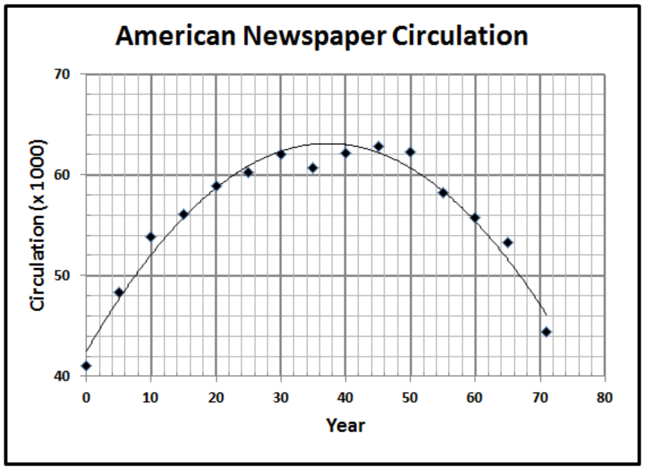
3. American newspaper circulation enjoyed continuous growth until the 1970's when it roughly leveled off then began a steady decline in the 1990's, presumably due to the internet.

Year	Circulation (in thousands)
1940	41.1
1945	48.4
1950	53.8
1955	56.1
1960	58.9
1965	60.3
1970	62.1
1975	60.7
1980	62.2
1985	62.8
1990	62.3
1995	58.2
2000	55.8
2005	53.3
2011	44.4

a) Consider 1940 to be year 0 and use regression to find a quadratic equation to model the data. Round the numbers in your equation to 3 decimal places.

b) It is clear in the table that circulation was at 50 (thousand) between 1945 and 1950, then again between 2005 and 2011. Use your equation to find the years, accurate to 1 decimal place.

c) Use your equation to predict the year the circulation will drop to 25 (thousand), accurate to 1 decimal place.



4. The number of inmates incarcerated in United States prisons is shown in the table.

Year	Prisoners (in thousands)
2005	1448
2006	1493
2007	1518
2008	1522
2009	1525
2010	1521
2011	1504

a) Consider 2005 to be year 0 and use regression to find a quadratic equation to model the data. Round the numbers in your equation to 3 decimal places.

b) Use your equation to predict the number of prisoners in 2016.

c) Use your equation to predict the year the U.S. prison population will be down to 1450, accurate to 1 decimal place.

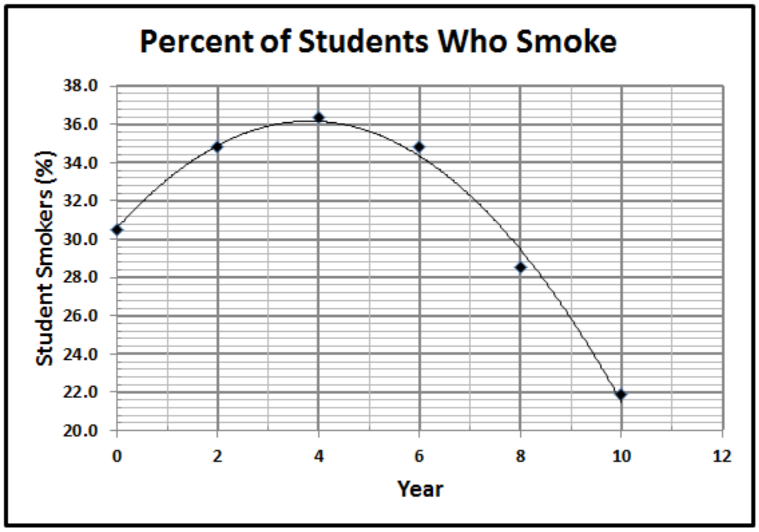
d) Find both years (past and future) for a prison population of 1400, accurate to 1 decimal place.

5. The percent of high school students who smoke is shown in the table.

Year	Percent
1993	30.5
1995	34.8
1997	36.4
1999	34.8
2001	28.5
2003	21.9

a) Consider 1993 to be year 0 and use regression to find a quadratic equation to model the data. Round the numbers in your equation to 3 decimal places.

b) Use your equation to predict the year when the percent of high school smokers will drop to 15%, accurate to 1 decimal place.



6. The value of Gentech stock is growing but beginning to level off. Use a parabola to model the data and consider the following questions:

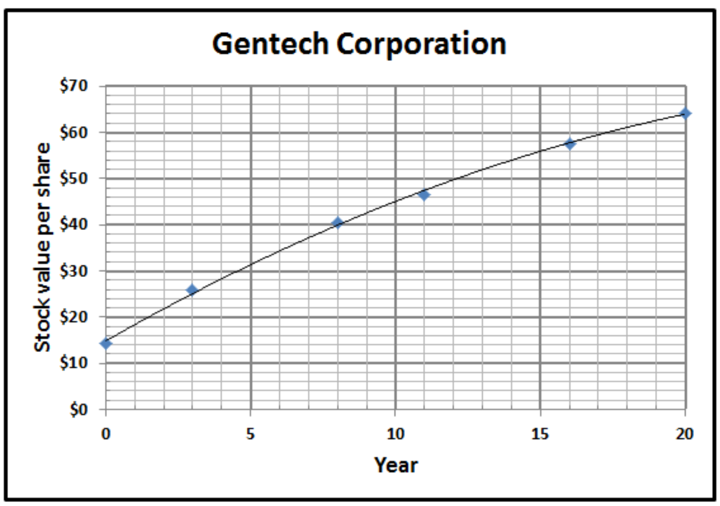
Year	Value
1982	\$14.20
1985	\$26.00
1990	\$40.40
1993	\$46.60
1998	\$57.60
2002	\$64.20

a) Consider 1982 to be year 0 and use regression to find a quadratic equation to model the data. Round the numbers in your equation to 3 decimal places.

b) Use your equation to predict the year when the stock will first reach a value of \$70 per share, accurate to 1 decimal place.

c) Use your equation to predict the value of the stock in 2020.

d) Use the vertex formula to find the vertex and interpret its meaning in context of the data.



7. The Egret population was dropping in a national park. In the year 2000, a biologist successfully suggested closing some trails that gave people access to their nesting sites. Use a parabola to model the data and consider the following questions:

Year	Population
2000	820
2003	582
2006	384
2008	276
2010	188
2013	98

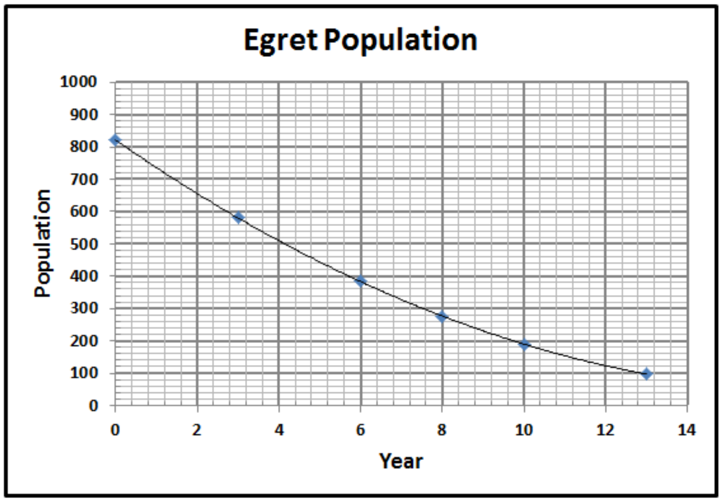
a) Consider 2000 to be year 0 and use regression to find a quadratic equation to model the data. Round the numbers in your equation to 3 decimal places.

b) Use your equation to predict the year when the population will drop below 30, accurate to 1 decimal place. (This is a bit of trick question that will make more sense if you are confused when you consider question (c))

c) Use the vertex formula to find the vertex and interpret its meaning in context of the data.

d) Use your equation to predict the population in 2015.

e) Use your equation to predict the year when the population will again reach 600, accurate to 1 decimal place.



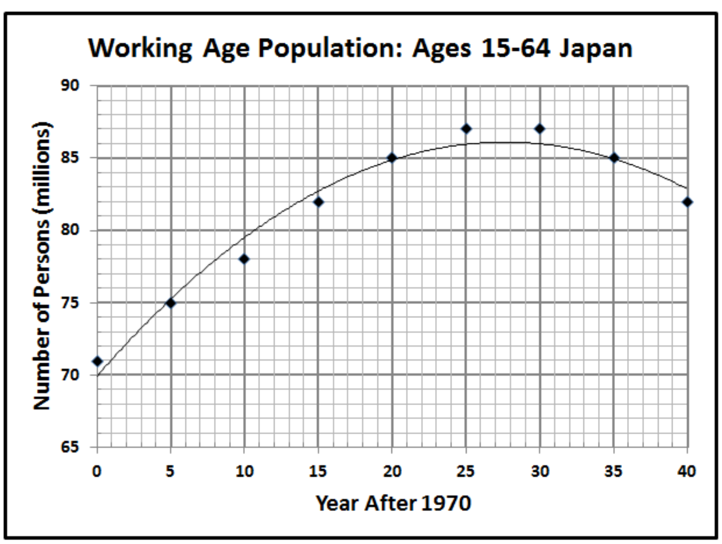
8. The population of working age people in Japan has seen a decline since about the year 2000.

a) Consider 1970 to be year 0 and use regression to find a quadratic equation to model the data. Round the numbers in your equation to 3 decimal places.

b) Use your equation to predict the year when the population of working age people will drop to 76 (million), accurate to 1 decimal place.

c) Use your equation to predict the population of working age people in 2016.

Year	Million Persons
1970	71
1975	75
1980	78
1985	82
1990	85
1995	87
2000	87
2005	85
2010	82



Chapter 3:

Power Relationships

The White Salmon River, Washington

River levels are one of many natural phenomenon that are accurately modeled by a power equation.

Rain and warm temperatures during spring snow melt will cause the river to rise.

Dry periods will cause the river level to fall.

In chapter 2 we focused on equations with an exponent (or power) of 2. In chapter 3 we go deeper into the same topic, considering equations of any decimal power.

Let's consider a river level application:

Example: Tracking River Levels

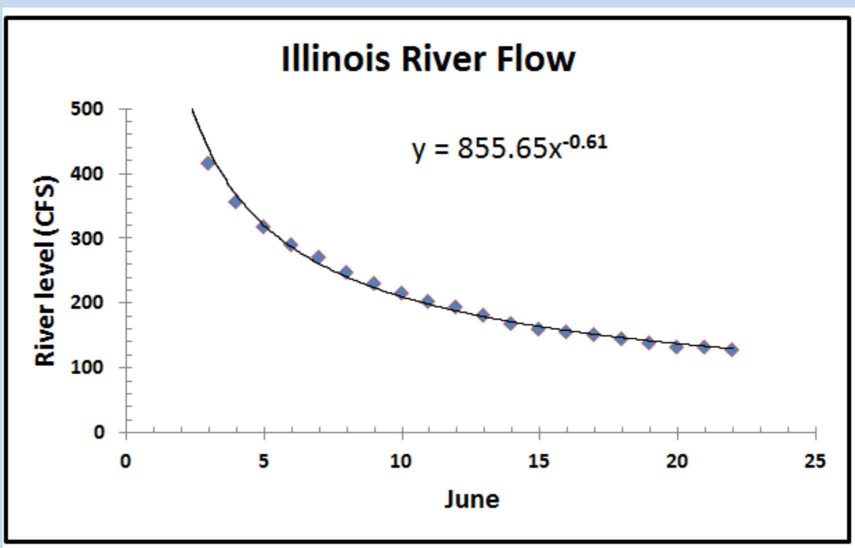
The Green Wall, Illinois River

Kayakers and fishermen regularly monitor river levels to gauge safety and fishing conditions. River levels are typically measured in cubic feet per second (CFS). The Illinois River is a favorite among skilled kayakers who watch the level in the late spring as the snow pack melts. It is passable down to 150 CFS in a small craft. The water data website listed below records updated levels for most of the rivers in the United States.
http://waterdata.usgs.gov/nwis

June	Level
3	416
4	357
5	318
6	291
7	271
8	248
9	230
10	216
11	203
12	194
13	181
14	169
15	160
16	155
17	152
18	146
19	140
20	133
21	132
22	128

Although the graph of the data from the table is curved like the parabola, the level will not come back up in the dry summer months, so a parabola will not be a good model.
This type of curve can be modeled by a **power equation**.

Notice the equation has a numerical exponent of -0.61.



Power equations take the form $y = ax^b$. We can use regression to find this type of equation.
Using the data in the table and "PwrReg", we get the same equation listed on the graph.
Notice the point (17,152) in the table. Enter 855.65*17^-.61 into your calculator as see if you get something close to 152.

Note: A rainstorm will of course spike this level back up but then it will begin dropping again, following a similar pattern.

Section 3.1: The Shape of a Power Equation

Let's start with a generic power equation and find its graph:

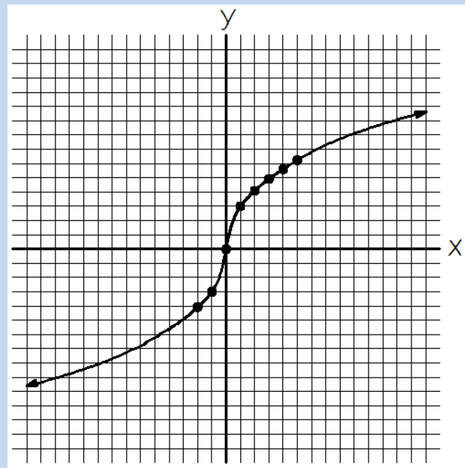
Example 3.1.1: Graphing a Power Equation

Find some ordered pairs then graph the equation: $y = 3x^{0.44}$

Solution:

You are free to pick any value for x in this power equation; there are many where x cannot be negative. Fortunately most real problems involve positive numbers. As with the creation of the quadratic formula in chapter 2; this is another very interesting discussion for those who have the time and interest.

A partial list of ordered pairs is shown in the table.



x	y
-2	-4.1
-1	-3
0	0
1	3
2	4.1
3	4.9
4	5.5
5	6.1

Remember to consider the order of operations when simplifying (PEMDAS).

$y = 3x^{0.44}$

$y = 3(2)^{0.44}$ replace x with 2

$y \approx 3(1.357)$ exponents first ($2^{.44}$ is entered as 2 ^ .44 in your calculator)

$y \approx 4.071$ multiplication

$y \approx 4.1$ rounded to the tenth place

We often choose whole numbered x's to make the calculation easier.

Can you replace x with 1.5 and get 3.6 for y? Notice (1.5,3.6) is also on the graph.

Can you replace x with $\frac{2}{3}$ and get 2.5 for y? Notice $\left(\frac{2}{3}, 2.5\right)$ is also on the graph.

Note: Notice that the negative exponent in the Illinois River example on the previous page produced a graph that decreased over time, while this graph, having a positive exponent, increased.

Now a practical problem:

Example 3.1.2: Graphing a Power Equation

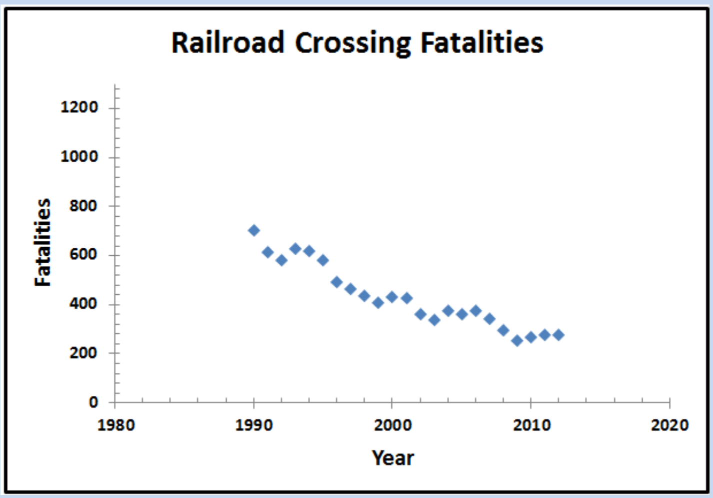
Consider the data regarding the number of automobile fatalities at railroad crossings in the United States since 1990.

1. Find and interpret the slope from 1990 to 1992.
2. Find and interpret the slope from 2008 to 2012.
3. Explain the change in the slopes you observe.
4. Graph the data.
5. Add a trend line and make a prediction for the number of fatalities in 2020.

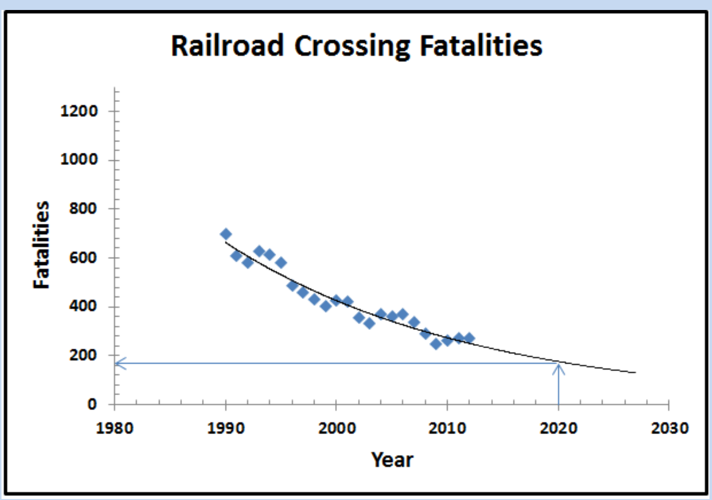
Year	Fatalities
1990	698
1991	608
1992	579
1993	626
1994	615
1995	579
1996	488
1997	461
1998	431
1999	402
2000	425
2001	421
2002	357
2003	334
2004	372
2005	359
2006	369
2007	339
2008	290
2009	249
2010	261
2011	271
2012	271

Solution:

1. $\frac{579-698}{1992-1990} = \frac{-119}{2} = \frac{-59.5}{1}$, meaning an average **decrease** of about 60 fatalities per year during this period of time.
2. $\frac{271-290}{2012-2008} = \frac{-19}{4} = \frac{-4.75}{1}$, meaning an average **decrease** of about 5 fatalities per year during this period of time.
3. Perhaps a change in legislation or the prevalence of safety technology caused the steeper decline in the early 90's. The rate of decline would have to level off however since we can never avoid auto accidents entirely. Power equations are excellent models for trends that level off.
4. See graph.



5. The trend would indicate that there will be about 160 fatalities in 2020.



Section 3.1: Problem Set

1. The American debt per capita is shown in the chart. This is the total debt divided among every man, woman and child in America … wow!

Year	Debt (thousands)
2010	$39.88
2011	$45.09
2012	$48.60
2013	$52.09
2014	$52.69

   a) Make a graph of the data large enough to include the year 2018 (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
   b) Find the slope between 2010 and 2012. Explain the meaning of the slope in context.
   c) Find the slope between 2013 and 2014. Explain the meaning of the slope in context.
   d) Add a trend line to the graph.
   e) By reading the graph, estimate the per capita debt that might be expected in 2018.

2. The table shows the gallons per minute (GPM) that various tank-less water heaters can produce based on the degree that the temperature must be raised. Consider the NR98 model for the questions below.

   a) Make a graph of the data large enough to include 20° (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
   b) Find the slope between 30° and 40°. Explain the meaning of the slope in context.
   c) Find the slope between 70° and 80°. Explain the meaning of the slope in context.
   d) Add a trend line to the graph.
   e) By reading the graph, estimate the GPM the NR98 will produce at 20°.

Temp Rise (°F)	Gallons per Minute (GPM)					
	NR111 (NC250) Series	NRC111 (NCC199) Series	NR98 (NC199) Series	NR83 Series	NR71 Series	NR66 Series
30	11.1	11.1	9.8	8.3	7.1	6.6
35	11.1	10.6	9.2	8.3	7.1	6.6
40	10.5	9.3	8.4	7.6	7.1	5.7
45	9.3	8.4	7.5	6.7	6.3	5.3
50	8.4	7.4	6.7	6.1	5.8	4.6
55	7.6	6.8	6.1	5.5	5.2	4.2
60	7.0	6.2	5.6	5.0	4.8	3.8
65	6.5	5.8	5.2	4.7	4.4	3.5
70	6.0	5.3	4.8	4.3	4.1	3.3
75	5.6	5.0	4.5	4.0	3.8	3.1
80	5.3	4.6	4.2	3.8	3.6	2.9
85	4.9	4.4	4.0	3.6	3.4	2.7
90	4.7	4.1	3.7	3.4	3.2	2.6
95	4.4	3.9	3.5	3.2	3.0	2.4
100	4.2	3.7	3.4	3.0	2.9	2.3

3. The table shows the natural gas flow in thousands of BTU's/hour for different pipe lengths and diameters. The longer the pipe the more inhibited the flow of natural gas because of the increase in friction. Consider the 3" pipe for the questions below.

   a) Make a graph of the data (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
   b) Find the slope between 30 and 40 feet. Explain the meaning of the slope in context.
   c) Find the slope between 100 and 125. Explain the meaning of the slope in context.
   d) Add a trend line to the graph.
   e) By reading the graph, estimate the BTU's per hour for a 250 foot pipe.

Natural Gas Piping Size Chart (From Aaladin)									
Length of Pipe in Feet	Size of Pipe in Inches								
	1/2"	3/4"	1"	1-1/4"	1-1/2"	2"	2-1/2"	3"	4"
10	108	230	387	793	1237	2259	3640	6434	
20	75	160	280	569	877	1610	2613	5236	9521
30	61	129	224	471	719	1335	2165	4107	7859
40	52	110	196	401	635	1143	1867	3258	6795
50	46	98	177	364	560	1041	1680	2936	6142
60	42	89	159	336	513	957	1559	2684	5647
70	38	82	149	317	476	896	1447	2492	5250
80	36	76	140	239	443	840	1353	2315	4900
90	33	71	133	275	420	793	1288	2203	4667
100	32	68	126	266	411	775	1246	2128	4518
125	28	60	117	243	369	700	1143	1904	4065
150	25	54	105	215	327	625	1008	1689	3645
175	23	50	93	196	303	583	993	1554	3370
200	22	47	84	182	280	541	877	1437	3160
300	17	37	70	145	224	439	686	1139	2539

4. Collisions at between trains and automobiles have declined since the 1990's, presumably due to increased safety measures.

   a) Make a graph of the data large enough to include the year 2020 (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
   b) Find the slope between 1991 and 1993. Explain the meaning of the slope in context.
   c) Find the slope between 2006 and 2010. Explain the meaning of the slope in context.
   d) Add a trend line to the graph.
   e) By reading the graph, estimate the number of collisions that might be expected in 2020.

Year	Collisions
1990	5388
1991	4910
1992	4892
1993	4979
1994	4633
1995	4257
1996	3865
1997	3508
1998	3489
1999	3502
2000	3237
2001	3077
2002	2977
2003	3077
2004	3057
2005	2936
2006	2776
2007	2429
2008	1934
2009	2052
2010	2062
2011	1960

5. The level in cubic feet per second(CFS) is shown for the White Salmon River for part of the month of May.

May	Level
8	744
9	645
10	558
11	492
12	442
13	398
14	367
15	342
16	324
17	302

a) Make a graph of the data large enough to include May 22nd (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
b) Find the slope between May 8th and 10th. Explain the meaning of the slope in context.
c) Find the slope between May 12th and 17th. Explain the meaning of the slope in context.
d) Add a trend line to the graph.
e) By reading the graph, estimate the level that a kayaker could expect on May 22nd.
f) By reading the graph, estimate the day the level will drop to 250 CFS.

6. The temperature of a closed car is shown over time on a day that is 80°F.

Minutes	Temperature
1	80
10	99
20	109
30	114
40	118
50	120
60	123

a) Make a graph of the data (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
b) Find the slope between 10 and 20 minutes. Explain the meaning of the slope in context.
c) Find the slope between 30 and 40 minutes. Explain the meaning of the slope in context.
d) Add a trend line to the graph.
e) If 105° is considered unsafe, read the graph to estimate the number of minutes it takes to reach 105°.

7. The population of Americans (in thousands) with no education is shown in the table.

Year	Population
1970	1569
1975	1293
1980	1068
1985	898
1990	765
1995	675
2000	621
2005	563
2010	480

a) Make a graph of the data large enough to include the year 2020 (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
b) Find the slope between 1980 & 1985. Explain the meaning of the slope in context.
c) Find the slope between 2000 & 2010. Explain the meaning of the slope in context.
d) Add a trend line to the graph
e) By reading the graph, estimate the population of uneducated Americans that might be expected in 2020.

8. The average concentration of PCB's in lake trout is shown in the table as a function of age. The EPA has classified PCB's as probable human carcinogens.

Year	PCB Conc.
1	1.0
2	1.9
4	4.2
6	7.2
8	11.8

   a) Make a graph of the data large enough to include 9 years (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
   b) Find the slope between age 2 & 4. Explain the meaning of the slope in context.
   c) Find the slope between age 6 & 8. Explain the meaning of the slope in context.
   d) Add a trend line to the graph.
   e) Estimate the PCB concentration you might expect in a 9 year old lake trout.

9. There is a relationship between the height and the volume of wood in a tree.

Height (ft)	Volume (ft^3)
12	82
20	180
26	267
34	395
40	501
50	691
60	898
75	1235

   a) Make a graph of the data large enough to show 1800 ft^3 (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
   b) Find the slope between heights 12 & 20. Explain the meaning of the slope in context.
   c) Add a trend line to the graph.
   d) Estimate the volume you might expect for a 55 foot tree.
   e) Estimate the height it would have to reach to contain 1800 ft^3.

10. Consider the price list for rectangular tarps of different sizes.

   a) Calculate the area and price/sqft.
   b) Make a graph of the data (use graph paper, label completely, and use area for the independent(x) and price per square foot for the dependent(y) variables).
   c) Add a trend line to the graph.
   d) Use your trend line to estimate the price per square foot and calculate the price (to the nearest dollar) for the remaining tarps in the table.

Dimensions	Area	Price	Price Per Square Foot
5' x 7'		$4	
6' x 8'		$5	
8' x 10'		$8	
10' x 12'		$11	
12' x 14'		$15	
16' x 20'		$26	
20' x 24'			
24' x 30'			

Section 3.2: Finding Power Equations

In this section we learn how to find a power equation from ordered pairs using regression.

As you may have noticed at this point in the course, the equation gives you more accurate predictive power than the graph and it is easy to find an ordered pair that is beyond the range of the graph.

You may also have noticed, however, that the equation takes more skill to use.

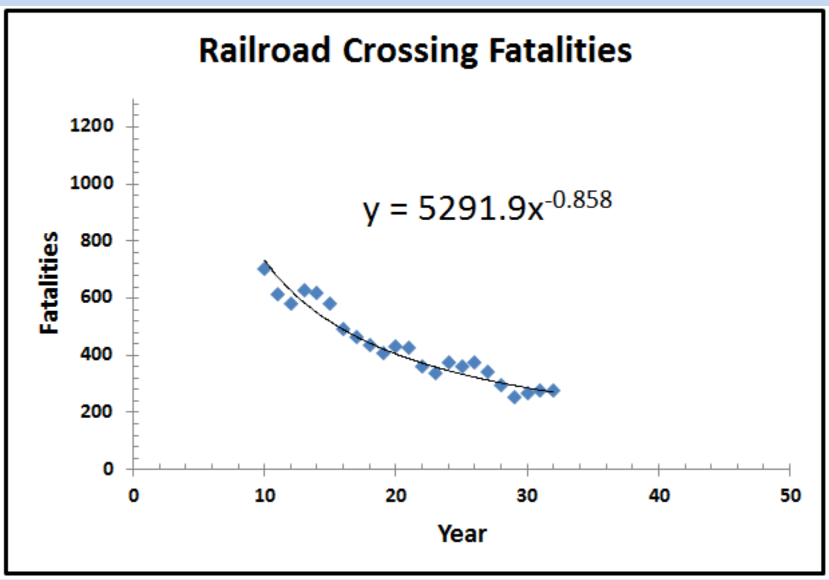
Reconsider the practical problem on railroad crossing fatalities introduced in section 3.1:

Example 3.2.1: Finding a Power Equation

Consider the data regarding the number of automobile fatalities at railroad crossings in the United States since 1990.
Use the regression feature of your graphing calculator to find an equation to model the data. Then use the equation to predict the number of fatalities there will be in 2017.

Year	Fatalities
1990	698
1991	608
1992	579
1993	626
1994	615
1995	579
1996	488
1997	461
1998	431
1999	402
2000	425
2001	421
2002	357
2003	334
2004	372
2005	359
2006	369
2007	339
2008	290
2009	249
2010	261
2011	271
2012	271

Solution:
For simplicity we can consider 1990 to be year 10. Enter the 23 ordered pairs into your calculator, use (PwrReg) and see if you agree with the equation shown on the graph.



Now, if 1990 is year 10 then 2017 would mean x = 37.

$y = 5291.9x^{-0.858}$

$y = 5291.9(37)^{-0.858}$ replace x with 37

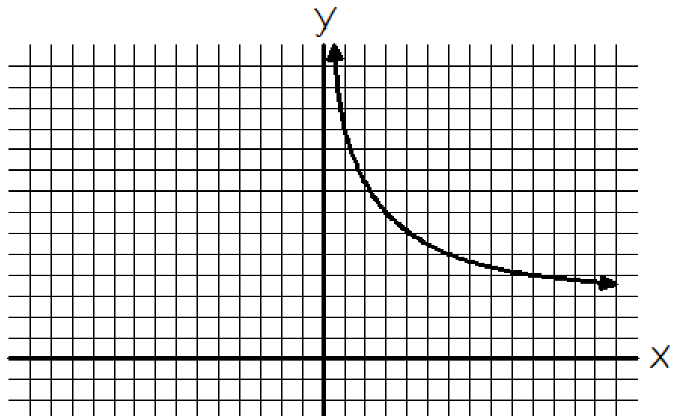
$y = 5291.9(.045132)$ exponents first

$y = 238.8$ multiplication last

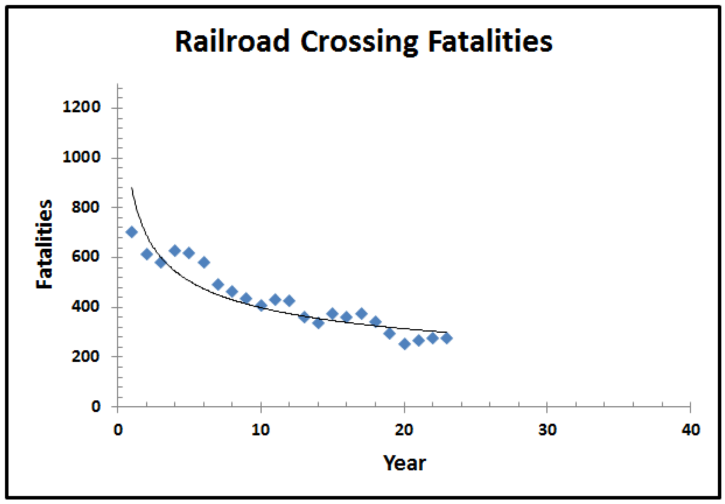
Final Answer: If the pattern continues to decline in a similar fashion, there will be approximately 239 fatalities at railroad crossings in 2017.

As we have seen previously, choosing 1990 as year 10 makes the numbers in the equation easier to work with, but there are some deeper considerations.

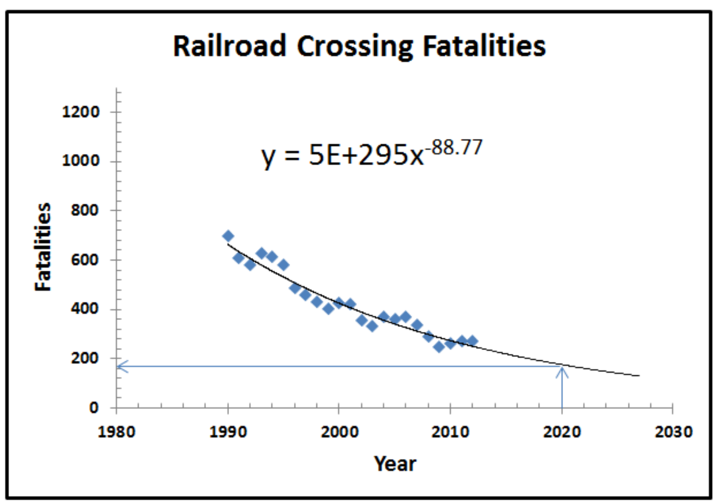
1. Choosing 1990 as year 0, as we have done in previous chapters, <u>will not work with power equations</u>. The y-coordinate of a power equation with a negative exponent goes off to infinity at x = 0 as shown on the graph.



2. If we had chosen 1990 as year 1 regression would have worked but would have produced a much steeper curve that is not a very good model for the data.



3. The best model is found entering the years as they are but the equation is so large that even calculators cannot handle it. The coefficient in front of x is scientific notation for 5 x 10^{295}, which is a 5 followed by 295 zeroes.



You will not have to concern yourself with these considerations in this chapter, since the questions will provide any direction you need. This must be understood, however, if you are to use power equations to model relationships that you come across in life … which is the whole point of this course.

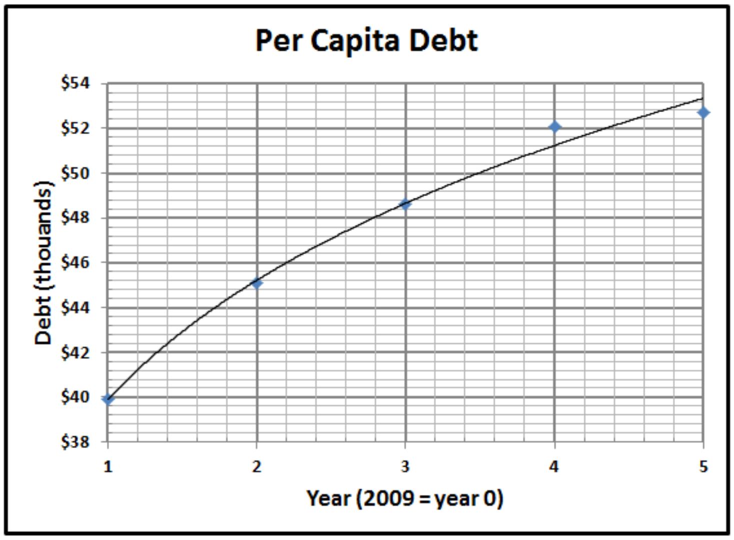
Section 3.2: Problem Set

1. The American debt per capita is shown in the chart. This is the total debt divided among every man, woman and child in America … wow!

Year	Debt (thousands)
2010	\$39.88
2011	\$45.09
2012	\$48.60
2013	\$52.09
2014	\$52.69

   a) Consider 2009 to be year 0 and use regression to find a power equation to model the data. Round the numbers in your equation to 3 decimal places.

   b) Use your equation to predict the debt in 2016.

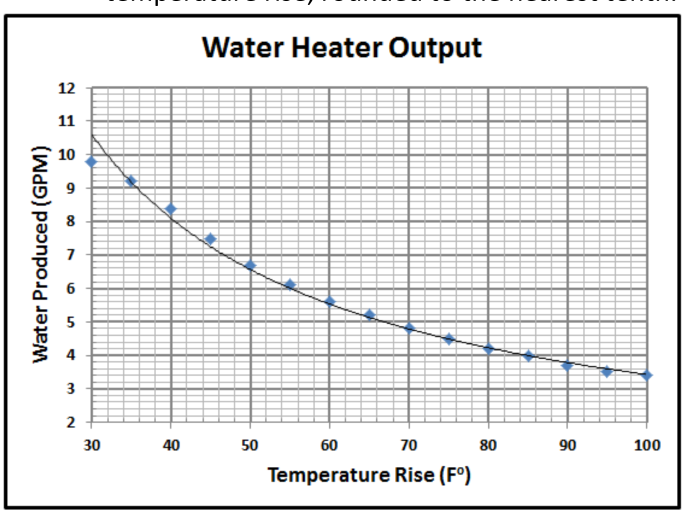


2. The table shows the gallons per minute (GPM) that various tank-less water heaters can produce based on the degree that the temperature must be raised.

   a) Use regression to find a power equation for the NRC 98 Series for the GPM as a function of the temperature. Round the numbers in your equation to 3 decimal places.

   b) Use your equation to predict the GPM for a 20° temperature rise, rounded to the nearest tenth.

Temp Rise (°F)	Gallons per Minute (GPM)					
	NR111 (NC250) Series	NRC111 (NCC199) Series	NR98 (NC199) Series	NR83 Series	NR71 Series	NR66 Series
30	11.1	11.1	9.8	8.3	7.1	6.6
35	11.1	10.6	9.2	8.3	7.1	6.6
40	10.5	9.3	8.4	7.6	7.1	5.7
45	9.3	8.4	7.5	6.7	6.3	5.3
50	8.4	7.4	6.7	6.1	5.8	4.6
55	7.6	6.8	6.1	5.5	5.2	4.2
60	7.0	6.2	5.6	5.0	4.8	3.8
65	6.5	5.8	5.2	4.7	4.4	3.5
70	6.0	5.3	4.8	4.3	4.1	3.3
75	5.6	5.0	4.5	4.0	3.8	3.1
80	5.3	4.6	4.2	3.8	3.6	2.9
85	4.9	4.4	4.0	3.6	3.4	2.7
90	4.7	4.1	3.7	3.4	3.2	2.6
95	4.4	3.9	3.5	3.2	3.0	2.4
100	4.2	3.7	3.4	3.0	2.9	2.3



3. The table shows the natural gas flow in thousands of BTU's/hour for different pipe lengths and diameters. The longer the pipe the more inhibited the flow of natural gas because of the increase in friction.

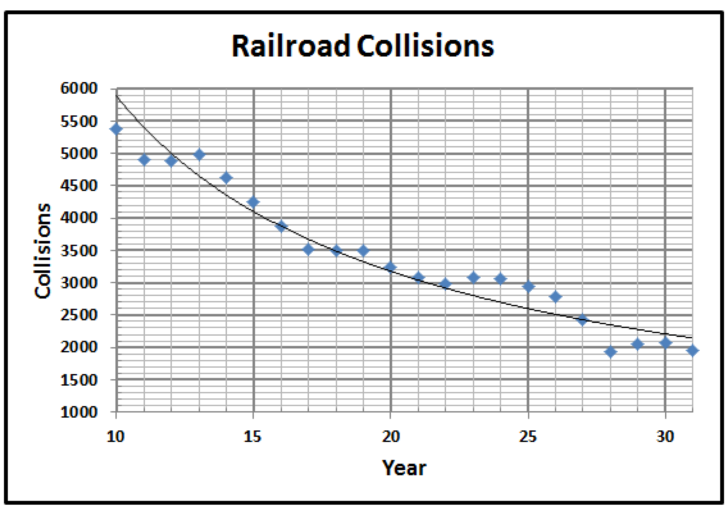
   a) Considering the 3 inch pipe and use regression to find a power equation where the flow is a function of the pipe length. Round the numbers in your equation to 3 decimal places.
   b) Use your equation to predict the flow rate for 250 feet of pipe, rounded to the nearest whole number.

Natural Gas Piping Size Chart (From Aaladin)									
Length of Pipe in Feet	Size of Pipe in Inches								
	1/2"	3/4"	1"	1-1/4"	1-1/2"	2"	2-1/2"	3"	4"
10	108	230	387	793	1237	2259	3640	6434	
20	75	160	280	569	877	1610	2613	5236	9521
30	61	129	224	471	719	1335	2165	4107	7859
40	52	110	196	401	635	1143	1867	3258	6795
50	46	98	177	364	560	1041	1680	2936	6142
60	42	89	159	336	513	957	1559	2684	5647
70	38	82	149	317	476	896	1447	2492	5250
80	36	76	140	239	443	840	1353	2315	4900
90	33	71	133	275	420	793	1288	2203	4667
100	32	68	126	266	411	775	1246	2128	4518
125	28	60	117	243	369	700	1143	1904	4065
150	25	54	105	215	327	625	1008	1689	3645
175	23	50	93	196	303	583	993	1554	3370
200	22	47	84	182	280	541	877	1437	3160
300	17	37	70	145	224	439	686	1139	2539

4. Collisions at between trains and automobiles have declined since the 1990's, presumably due to increased safety measures.

   a) Consider 1990 to be year 10 and use regression to find a power equation to model the data. Round the numbers in your equation to 3 decimal places.
   b) Use your equation to predict the number of collisions in 2025, round to the nearest whole number.

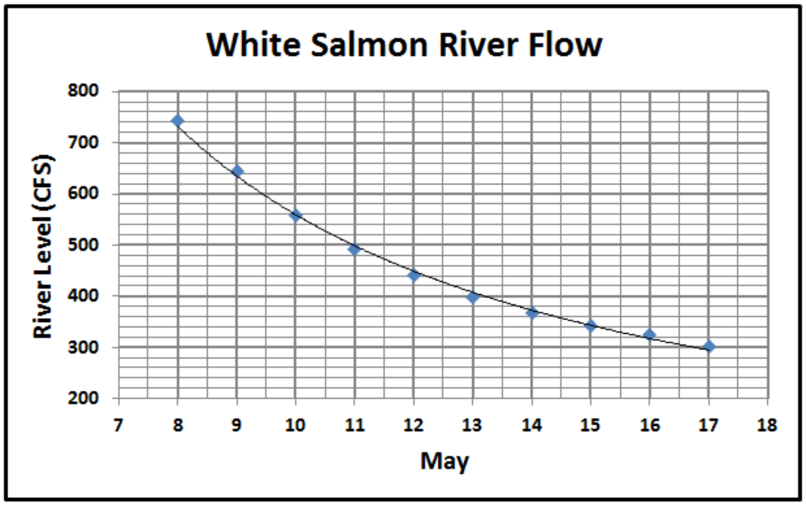
Year	Collisions
1990	5388
1991	4910
1992	4892
1993	4979
1994	4633
1995	4257
1996	3865
1997	3508
1998	3489
1999	3502
2000	3237
2001	3077
2002	2977
2003	3077
2004	3057
2005	2936
2006	2776
2007	2429
2008	1934
2009	2052
2010	2062
2011	1960



5. The level (in CFS) is shown for the White Salmon River for part of the month of May.

   a) Use regression to find a power equation to model the data. Round the numbers in your equation to 3 decimal places.
   b) Use your equation to predict the level on May 24th, round to the nearest whole number.

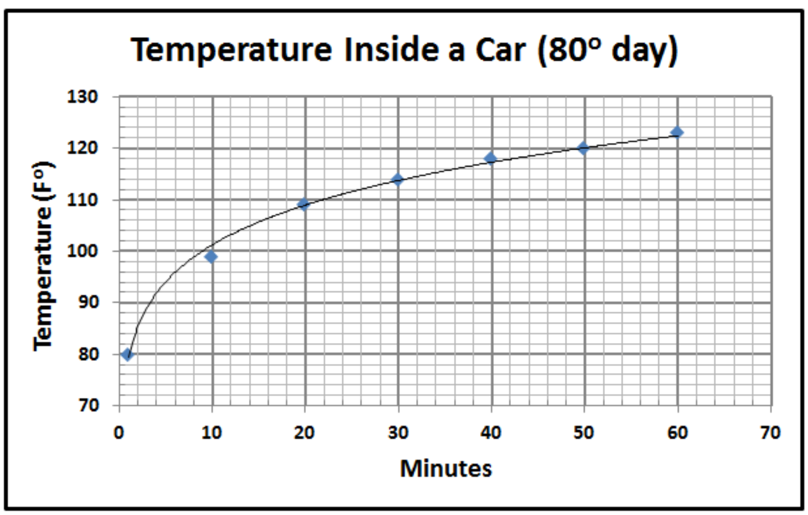
May	Level
8	744
9	645
10	558
11	492
12	442
13	398
14	367
15	342
16	324
17	302



6. The temperature of a closed car is shown over time on a day that is 80°F.

   a) Use regression to find a power equation to model the data. Round the numbers in your equation to 3 decimal places.
   b) Use your equation to predict the temperature at 5 minutes, round to the nearest tenth.

Minutes	Temperature
1	80
10	99
20	109
30	114
40	118
50	120
60	123

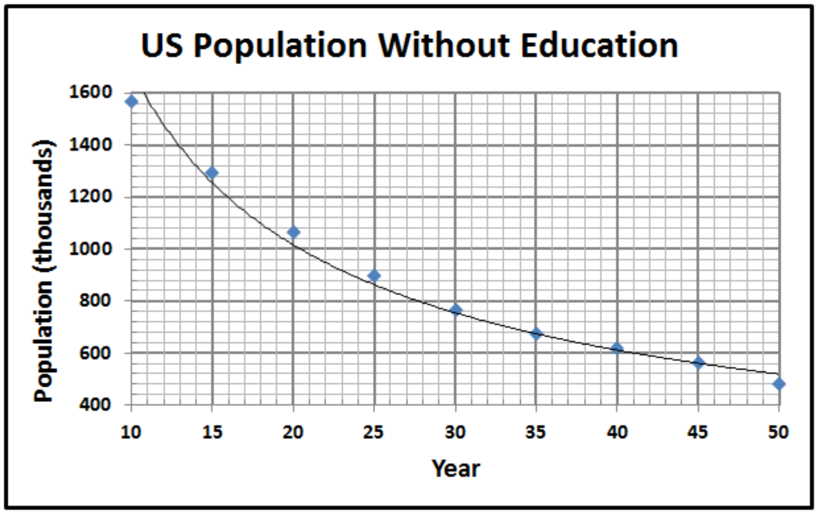


7. The population of Americans (in thousands) with no education is shown in the table.

Year	Population
1970	1569
1975	1293
1980	1068
1985	898
1990	765
1995	675
2000	621
2005	563
2010	480

a) Consider 1970 as year 10 and use regression to find a power equation to model the data. Round the numbers in your equation to 3 decimal places.

b) Use your equation to predict the population in 2025, round to the nearest whole number.

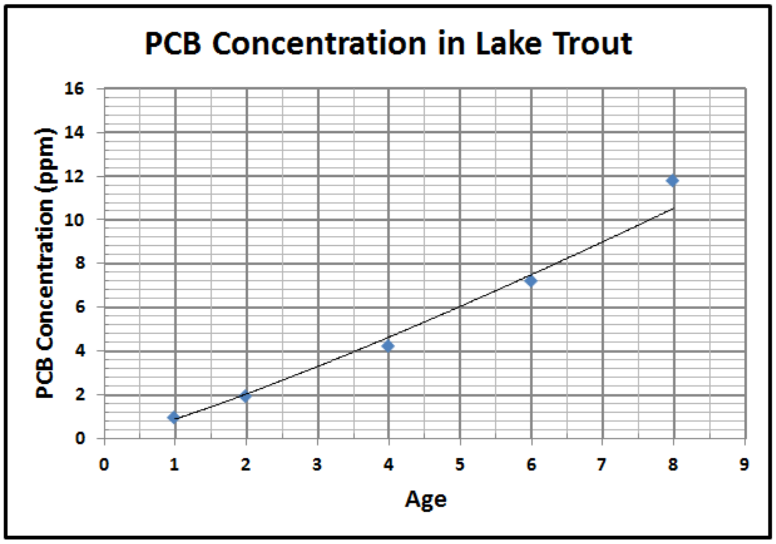


8. The average concentration of PCB's, measured in parts per million (PPM), in lake trout is shown in the table as a function of age. The EPA has classified PCB's as probable human carcinogens.

Year	PCB's (PPM)
1	1.0
2	1.9
4	4.2
6	7.2
8	11.8

a) Use regression to find a power equation to model the data. Round the numbers in your equation to 3 decimal places.

b) Use your equation to predict the PCB concentration that you could expect in a 10 year old trout, round to the nearest tenth.

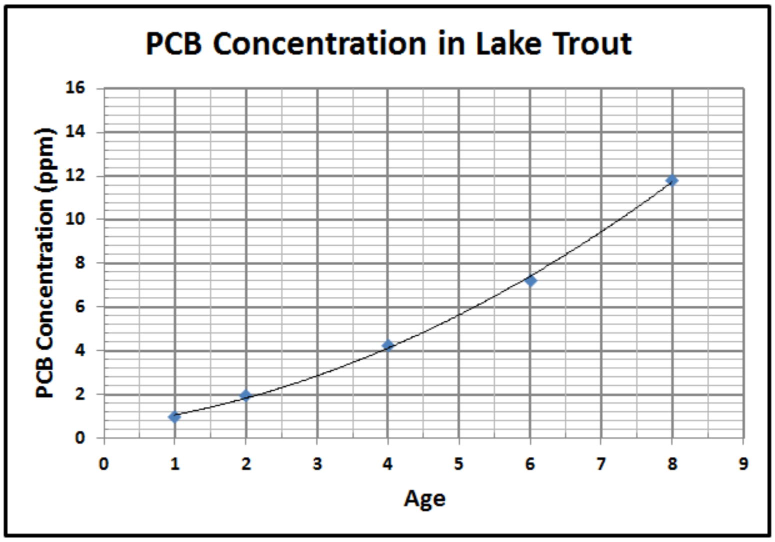


9. Since we now have quadratic and power models that both accommodate curved data, reconsider the previous question.

Year	PCB's (PPM)
1	1.0
2	1.9
4	4.2
6	7.2
8	11.8

a) Notice the power curve in the previous question is not a great fit to the data (compare the fit of a quadratic curve below to the power function in the previous problem). Use regression to find a <u>quadratic</u> equation to model the data. Round the numbers in your equation to 3 decimal places.

b) Use your equation to make a better prediction for the PCB concentration that you could expect in a 10 year old trout, round to the nearest tenth.

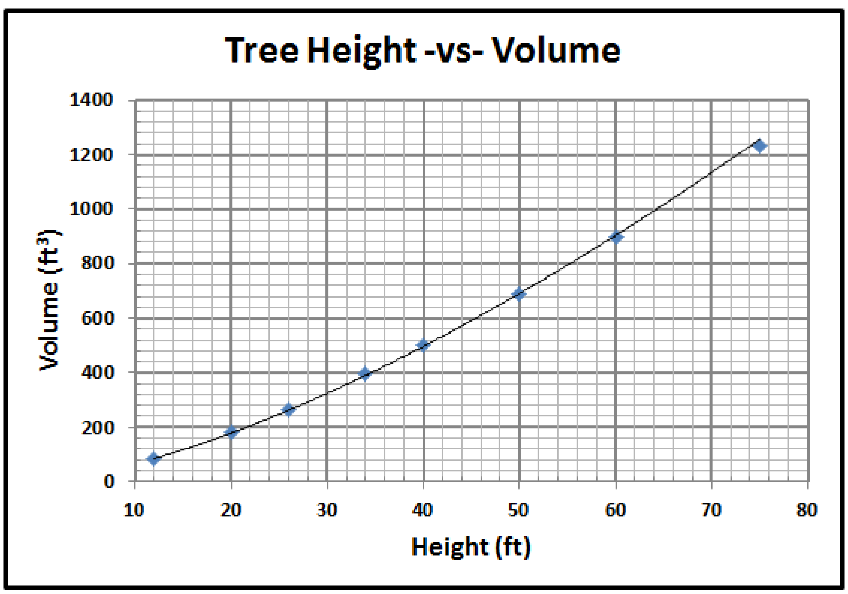


10. There is a relationship between the height and the volume of wood in a tree.

Height (ft)	Volume (ft^3)
12	82
20	180
26	267
34	395
40	501
50	691
60	898
75	1235

a) Use regression to find a power equation to model the data. Round the numbers in your equation to 3 decimal places.

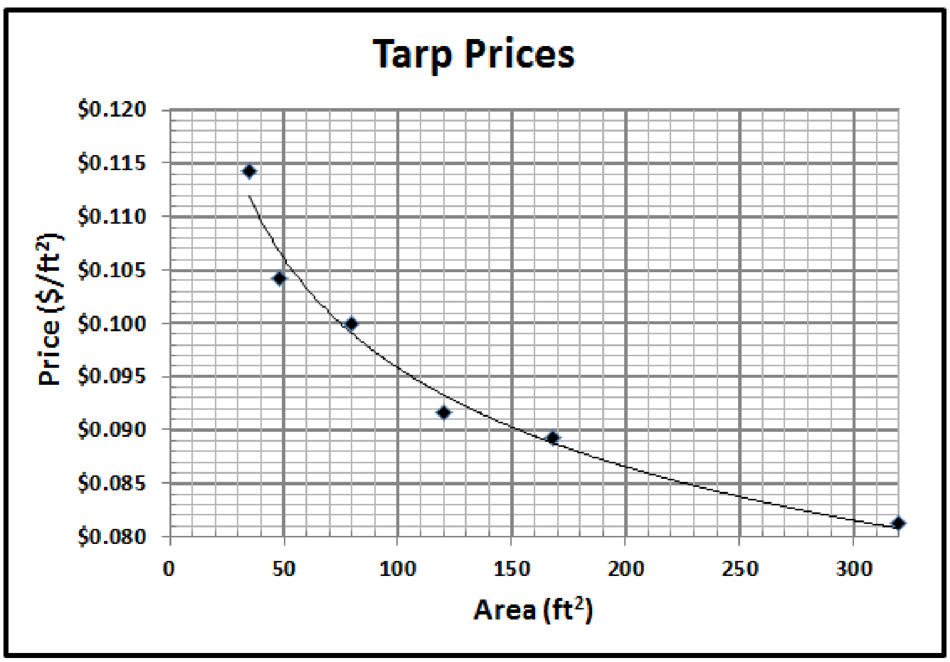
b) Use your equation to predict the volume of a 90 foot tree, round to the nearest whole number.



11. Consider the price list for rectangular tarps of different sizes.

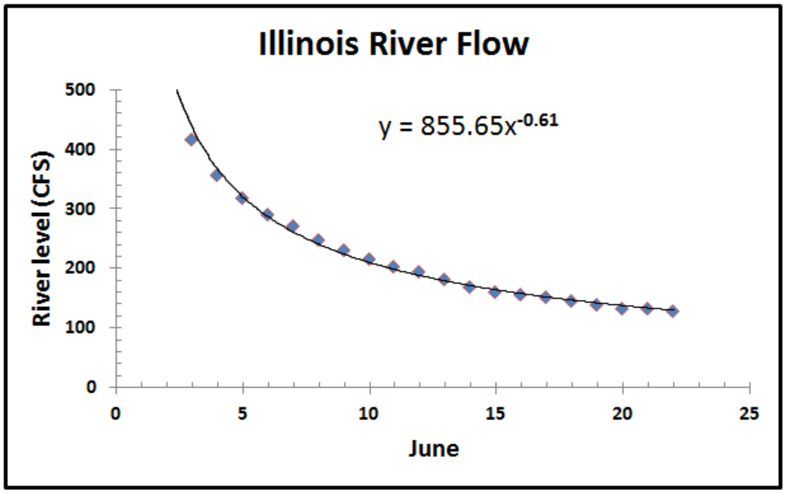
   a) Use regression to find a power equation to model the data using area as the x variable and price per square foot as the y variable. Round the numbers in your equation to 3 decimal places.
   b) Use your equation to determine the right price per square foot (rounded to 3 decimal places) and price for the 2 larger tarps (rounded to the nearest dollar) in the table.

Dimensions	Area	Price	Price Per Square Foot
5' x 7'		\$4	
6' x 8'		\$5	
8' x 10'		\$8	
10' x 12'		\$11	
12' x 14'		\$15	
16' x 20'		\$26	
20' x 24'			
24' x 30'			



Section 3.3: Using Power Equations

As in section 2.3 we now turn our attention to the complex and practical skill of finding an "x" that will produce a "y" that we are interested in.



Refer back to the river level example in the introduction to chapter 3; suppose the fishing gets good below 100 CFS on the Illinois River. We would of course want to know what day this would happen. We could guess values for x until we figure it out. This is not a bad idea since it does not require us to learn anything new.

$855.65(30)^{-0.61} \approx 107$ guess 30 … too high!

$855.65(35)^{-0.61} \approx 98$ guess 35 … too low!

$855.65(33)^{-0.61} \approx 101.4$ guess 33 … close!

$855.65(34)^{-0.61} \approx 99.6$ guess 34 … close!

There are 30 days in June so the answer looks to be late in the day on July 3rd. One of the beauties of mathematics lies in being able to figure things out accurately without guessing though!

How can we solve for "x" without guessing where $100 = 855.65x^{-0.61}$?

Solving simply means finding out what "x" is equal to. We need to get rid of the 855.65 and the -0.61.

When we knew "x", we used the order of operations (PEMDAS) to simplify. Now we need to "unwrap" "x", so we remove the numbers using inverse operations and in the reverse order (SADMEP).

$100 = 855.65x^{-0.61}$

$0.117 = x^{-0.61}$ remove the multiplication by dividing both sides by 855.65

Note: when an exponent is raised to another exponent, multiply the exponents to simplify. $(x^3)^2 = x^6$.

$0.117^{\left(\frac{1}{-0.61}\right)} = (x^{-0.61})^{\left(\frac{1}{-0.61}\right)}$ raise both sides to the $\frac{1}{-.61}$ power

$\mathbf{0.117^{\frac{1}{-.61}} = x}$ $\quad \mathbf{-0.61 * \frac{1}{-0.61} = 1}$

$x \approx 33.7$ enter .117 ^ (1/-0.61) into the calculator

We get a number between 33 & 34 as we expected from our guessing.

<u>Clearing an exponent:</u>

$$(x^a)^{\frac{1}{a}} = x$$

For a variable "x" with an exponent "a" … raise to the $\frac{1}{a}$ power to clear the exponent.

Section 3.3: Problem Set

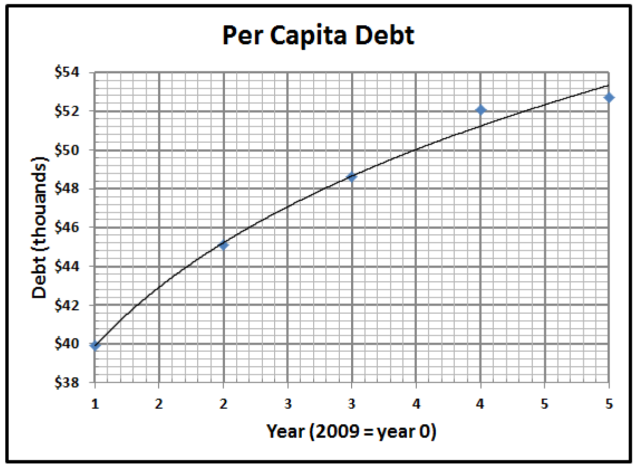
1. The American debt per capita is shown in the chart. This is the total debt divided among every man, woman and child in America … wow!

Year	Debt (thousands)
2010	$39.88
2011	$45.09
2012	$48.60
2013	$52.09
2014	$52.69

   a) Consider 2009 to be year 0 and use regression to find a power equation to model the data. Round the numbers in your equation to 2 decimal places.

   b) Use your equation to predict the debt in 2018.

   c) Use your equation to predict the year the debt will reach $65,000.



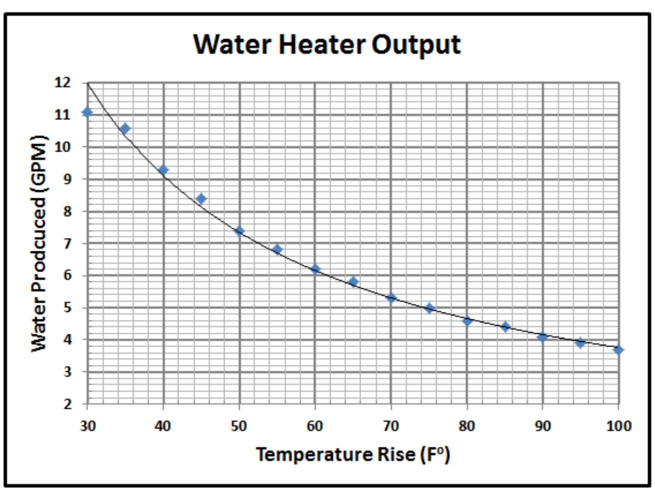
2. The table shows the gallons per minute (GPM) that various tank-less water heaters can produce based on the degree that the temperature must be raised.

Temp Rise (˚F)	Gallons per Minute (GPM)					
	NR111 (NC250) Series	NRC111 (NCC199) Series	NR98 (NC199) Series	NR83 Series	NR71 Series	NR66 Series
30	11.1	11.1	9.8	8.3	7.1	6.6
35	11.1	10.6	9.2	8.3	7.1	6.6
40	10.5	9.3	8.4	7.6	7.1	5.7
45	9.3	8.4	7.5	6.7	6.3	5.3
50	8.4	7.4	6.7	6.1	5.8	4.6
55	7.6	6.8	6.1	5.5	5.2	4.2
60	7.0	6.2	5.6	5.0	4.8	3.8
65	6.5	5.8	5.2	4.7	4.4	3.5
70	6.0	5.3	4.8	4.3	4.1	3.3
75	5.6	5.0	4.5	4.0	3.8	3.1
80	5.3	4.6	4.2	3.8	3.6	2.9
85	4.9	4.4	4.0	3.6	3.4	2.7
90	4.7	4.1	3.7	3.4	3.2	2.6
95	4.4	3.9	3.5	3.2	3.0	2.4
100	4.2	3.7	3.4	3.0	2.9	2.3

   a) Use regression to find a power equation for the NRC 111 Series for the GPM as a function of the temperature. Round the numbers in your equation to 2 decimal places.

   b) Use your equation to predict the GPM for a 25° temperature rise, rounded to the nearest tenth.

   c) Use your equation to predict temperature gain achieved at 8 GPM, rounded to the nearest tenth.

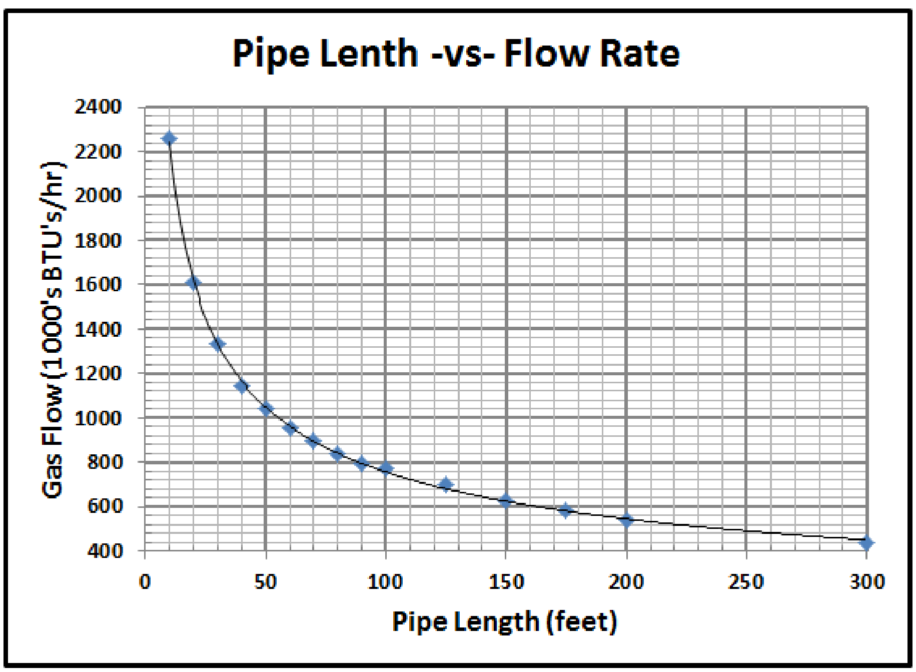


3. The table shows the natural gas flow in thousands of BTU's/hour for different pipe lengths and diameters. The longer the pipe the more inhibited the flow of natural gas because of the increase in friction.

   a) Considering the 2 inch pipe and use regression to find a power equation where the flow is a function of the pipe length. Round the numbers in your equation to 2 decimal places.
   b) Use your equation to predict the flow rate for 250 feet of pipe, rounded to the nearest whole number.
   c) Use your equation to find the maximum length of pipe allowable if you need 1,200,000 BTU's/hour of gas flow in a 2 inch pipe, rounded to the nearest tenth.

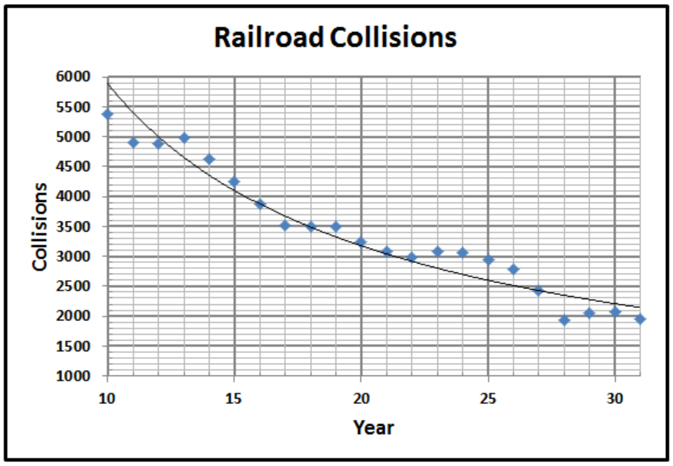
Natural Gas Piping Size Chart (From Aaladin)

Length of Pipe in Feet	Size of Pipe in Inches								
	1/2"	3/4"	1"	1-1/4"	1-1/2"	2"	2-1/2"	3"	4"
10	108	230	387	793	1237	2259	3640	6434	
20	75	160	280	569	877	1610	2613	5236	9521
30	61	129	224	471	719	1335	2165	4107	7859
40	52	110	196	401	635	1143	1867	3258	6795
50	46	98	177	364	560	1041	1680	2936	6142
60	42	89	159	336	513	957	1559	2684	5647
70	38	82	149	317	476	896	1447	2492	5250
80	36	76	140	239	443	840	1353	2315	4900
90	33	71	133	275	420	793	1288	2203	4667
100	32	68	126	266	411	775	1246	2128	4518
125	28	60	117	243	369	700	1143	1904	4065
150	25	54	105	215	327	625	1008	1689	3645
175	23	50	93	196	303	583	993	1554	3370
200	22	47	84	182	280	541	877	1437	3160
300	17	37	70	145	224	439	686	1139	2539



4. Collisions at between trains and automobiles have declined since the 1990's, presumably due to increased safety measures.

   a) Consider 1990 to be year 10 and use regression to find a power equation to model the data. Round the numbers in your equation to 2 decimal places.
   b) Use your equation to predict the year that the number of collisions will drop to 1900, rounded to the nearest tenth.
   c) Use your equation to predict the year that the number of collisions will drop to 1750, rounded to the nearest tenth.

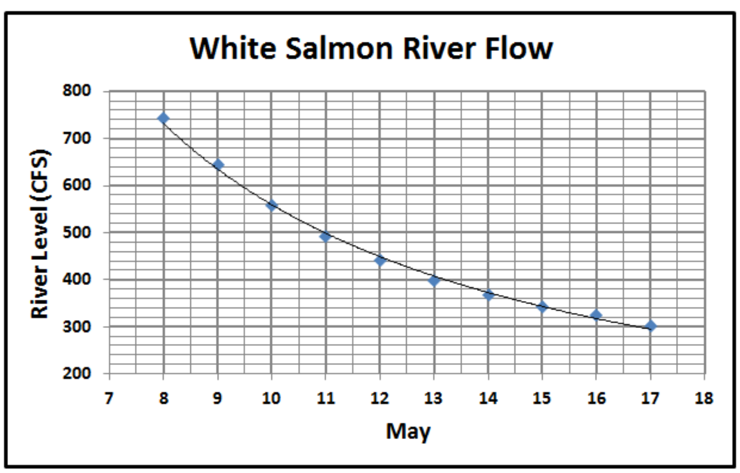


Year	Collisions
1990	5388
1991	4910
1992	4892
1993	4979
1994	4633
1995	4257
1996	3865
1997	3508
1998	3489
1999	3502
2000	3237
2001	3077
2002	2977
2003	3077
2004	3057
2005	2936
2006	2776
2007	2429
2008	1934
2009	2052
2010	2062
2011	1960

5. The level (in CFS) is shown for the White Salmon River for part of the month of May.

   a) Use regression to find a power equation to model the data. Round the numbers in your equation to 2 decimal places.
   b) Use your equation to find the day the level was 1000 CFS, rounded to the nearest tenth.
   c) Use your equation to find the day the level will drop to 250 CFS, rounded to the nearest tenth.

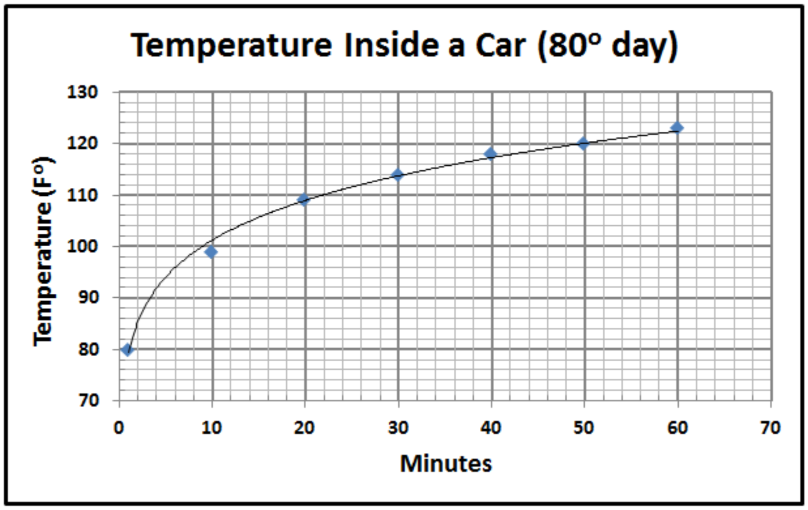
May	Level
8	744
9	645
10	558
11	492
12	442
13	398
14	367
15	342
16	324
17	302



6. The temperature of a closed car is shown over time on a day that is 80°F.

   a) Use regression to find a power equation to model the data. Round the numbers in your equation to 2 decimal places.
   b) Use your equation to find the number of minutes required to reach 90°, rounded to the nearest tenth.
   c) Use your equation to find the number of minutes required to reach 128°, rounded to the nearest tenth.

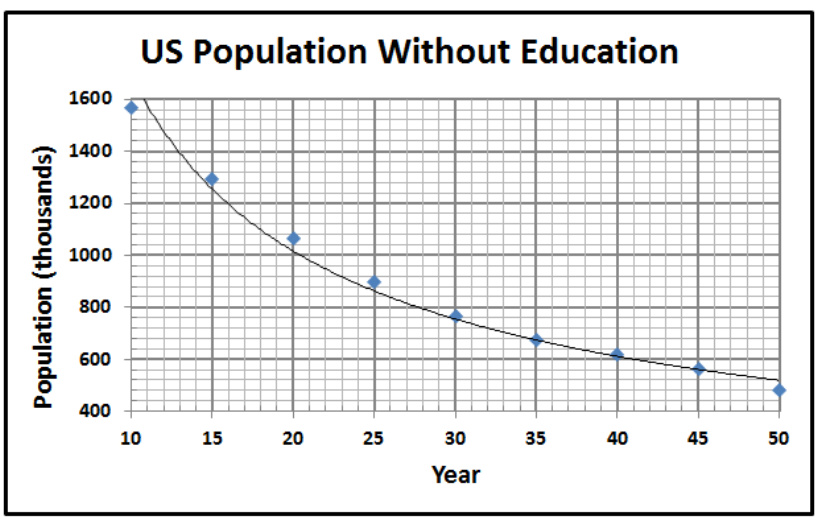
Minutes	Temperature
1	80
10	99
20	109
30	114
40	118
50	120
60	123



7. The population of Americans (in thousands) with no education is shown in the table.

   a) Consider 1970 as year 10 and use regression to find a power equation to model the data. Round the numbers in your equation to 2 decimal places.
   b) Use your equation to accurately give year the population was 600 (thousand), rounded to the nearest tenth.

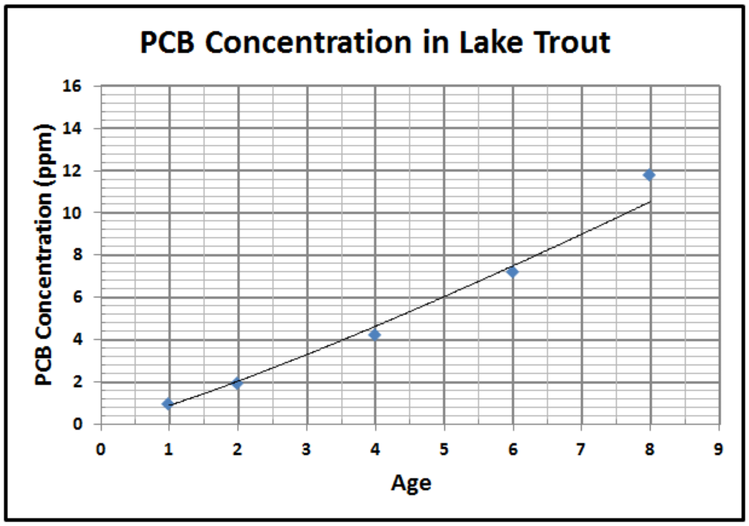
Year	Population
1970	1569
1975	1293
1980	1068
1985	898
1990	765
1995	675
2000	621
2005	563
2010	480



8. The average concentration of PCB's in lake trout is shown in the table as a function of age. The EPA has classified PCB's as a probable human carcinogen.

Year	PCB Conc.
1	1.0
2	1.9
4	4.2
6	7.2
8	11.8

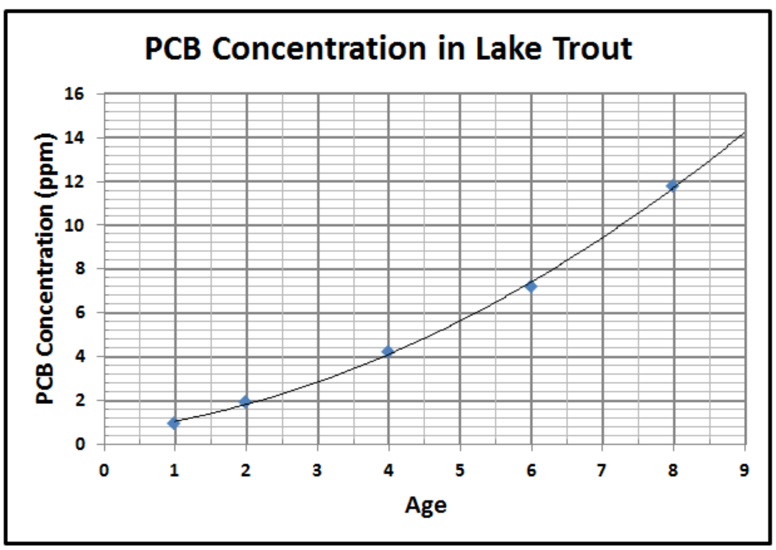
   a) Use regression to find a power equation to model the data. Round the numbers in your equation to 2 decimal places.
   b) Use your equation to estimate the age of a fish with a PCB concentration of 15 ppm, rounded to the nearest tenth.
   c) Use your equation to estimate the age of a fish with a PCB concentration of 6 ppm, rounded to the nearest tenth.



9. Since we now have quadratic and power models that both model curved data, reconsider the previous question.

Year	PCB Conc.
1	1.0
2	1.9
4	4.2
6	7.2
8	11.8

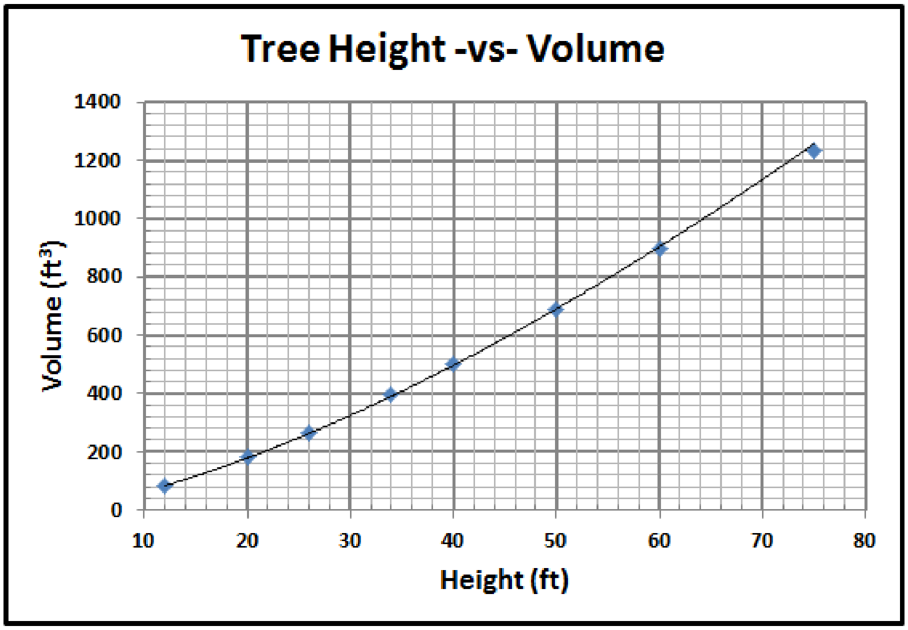
   a) Notice the power curve in the previous question is not that great of a fit to the data (compare the fit of a quadratic curve below to the power function above). Use regression to find a quadratic equation to model the data. Round the numbers in your equation to 2 decimal places.
   b) Use your quadratic equation to estimate the age of a fish with a PCB concentration of 15 ppm, rounded to the nearest tenth.



10. There is a relationship between the height and the volume of wood in a tree.

    a) Use regression to find a power equation to model the data. Round the numbers in your equation to 2 decimal places.
    b) Use your equation to find the height a tree should be to have a volume of 300 ft^3, rounded to the nearest tenth.
    c) Use your equation to find the height a tree should be to have a volume of 1600 ft^3, rounded to the nearest tenth.

Height (ft)	Volume (ft^3)
12	82
20	180
26	267
34	395
40	501
50	691
60	898
75	1235



Chapter 4:

Exponential Relationships

Oregon elk population

Animal populations are one of many aspects of nature modeled by exponential equations.

Favorable conditions allow populations to increase while cold winters or an increase in predators may cause the population to decrease.

In either case biologists can accurately model and monitor the size and health of a population with exponential equations.

In chapter 3 we extended our use of equations with exponents and considered equations with decimal powers. In this section we will consider a class of equations where the exponent is a variable. In most practical uses this variable represents time.

The birth of an exponential equation is easily seen with a basic application of percentages. A biologist who wants to know how many years it will take for a population of 1200 salmon growing in number at a rate of 8% per year to reach 10,000, will need an exponential equation.

Example: Tracking a Salmon Population

Sockeye salmon run

The population is first observed at 1200 salmon (year 0). At a growth rate of 8% per year there would be 1200 • 0.08 = 96 more salmon next year, or 1296 salmon. A shortcut to 1296 is to multiply 1200 • 1.08, since growing 8% means 8% more than 100%, or 108%.

year 1: 1200 • 1.08 = 1296
year 2: 1296 • 1.08 ≈ 1400
year 3: 1400 • 1.08 ≈ 1512
year 4: 1512 • 1.08 ≈ 1633

Year	Population
0	1200
1	1296
2	1400
3	1512
4	1633
5	1763
6	1904

There is a further shortcut to the population in year 4. Notice that 1200 is multiplied by 1.08 four times. This fact is easily expressed with the exponents we studied in chapters 2 & 3.
$1200 \bullet 1.08 \bullet 1.08 \bullet 1.08 \bullet 1.08 = 1200 \bullet 1.08^4$

Using this shortcut we can guess and check our way to a population of 10,000. Notice the exponent represents the time (in this case years) we want to know.

$1200 \bullet 1.08^{11} \approx 2798$ guess 11 … too low
$1200 \bullet 1.08^{17} \approx 4440$ guess 17 … too low
$1200 \bullet 1.08^{22} \approx 6525$ guess 22 … too low
$1200 \bullet 1.08^{30} \approx 12075$ guess 30 … too high
$1200 \bullet 1.08^{28} \approx 10353$ guess 28 … too high
$1200 \bullet 1.08^{27} \approx 9586$ The population will reach 10,000 sometime during year 28.

Exponential equations take the form $y = ab^x$; where "a" represents the initial population, "b" represents the growth rate (or decline rate), and "x" is the time.

Note: The growth rate of 8% was given in this problem. Notice it can be found by dividing the population of one year by the population of the previous year. If the year before the biologist counted 1111 salmon, the rate of change can be calculated to be $\frac{1200}{1111} \approx 1.08$.

Section 4.1: The Shape of an Exponential Equation

Let's look at a generic exponential equation and find its graph:

Example 4.1.1: Graphing an Exponential Equation

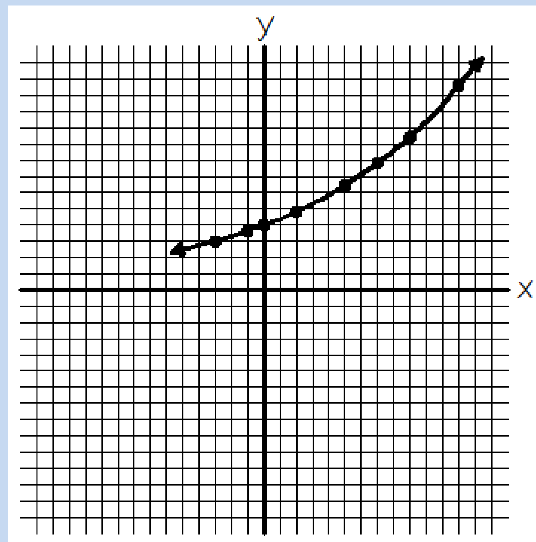
Find some ordered pairs then graph the equation: $y = 4 \bullet (1.1)^x$

Solution:

You are free to pick any value for "x" in the exponential equation; even negative numbers since they will represent the population back in time.

A partial list of ordered pairs is shown in the table.

x	y
-3	3.0
-1	3.6
0	4.0
2	4.8
5	6.4
7	7.8
9	9.4
12	12.6



Remember to consider the order of operations when simplifying (PEMDAS).

$y = 4\bullet(1.1)^x$	
$y = 4\bullet(1.1)^5$	replace x with 5
$y \approx 4\bullet(1.61)$	exponents first (1.1^5 is entered as 1.1 ^ 5 in your calculator)
$y \approx 6.44$	multiplication
$y \approx 6.4$	rounded to the tenth place

Note: Notice that this equation could represent a disease that is discovered in 4 patients and grows at a rate of 10% per year.

Let's look at an example where the population is decreasing:

Example 4.1.2: Moisture Content in Lumber

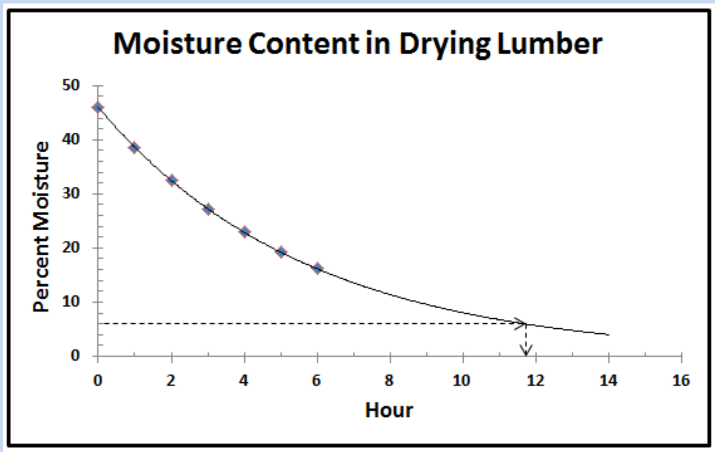
Lumber is routinely dried before use in residential applications to keep it from shrinking and twisting after installation.

Hour	% Moisture
0	46.0
1	38.6
2	32.5
3	27.3
4	22.9
5	19.2
6	16.2

1. Make a graph of the data which shows the moisture content in Douglas fir lumber in a dryer over a 6 hour period.
2. Add a trend line and make a prediction for the time required for the moisture content to reach 6%.
3. Find the growth rate from hour 1 to 2
4. Find the growth rate from hour 4 to 5

Solution:

1. Hour is the independent variable and percent is the dependent variable.
2. Graph indicates between hours 11 and 12.
3. $\frac{32.5}{38.6} \approx 0.842$ or about 84.2%
4. $\frac{19.2}{22.9} \approx 0.838$ or about 83.8%



Indeed the growth rate between any pair of consecutive points is around 84%. Each hour the lumber retains 84% of the moisture it had the previous hour.
Therefore the equation $y = 46 \bullet (0.84)^x$ would model this data.

Note: This is the essence of any exponential equation; there is a constant growth rate between each data point over a given period of time.
Notice that this data could also be thought of as a decrease in moisture of 16% per hour.

To simplify the language we will always speak of exponential data as having a **growth rate**, even if it is declining. In the previous example this can appear misleading since the moisture content was not growing.
The principle is simple if keep your focus on 100%. A population with a growth rate of 109% is understood to be increasing by 9%. A population with a growth rate of 87% is understood to be decreasing by 13%.

There is a subtle but important distinction between how we looked at the constant ***rate of change*** with linear relationships in chapter 1 and how we look at constant ***growth rate*** with exponential relationships in this chapter. For a linear equation the constant is ***added*** to the previous value whereas in an exponential equation the constant is ***multiplied*** by the previous value.

Consider the populations of 24 frogs, increasing as indicated in the chart below over a 3 year period.

Year	linear	exponential
0	64	64
1	80	80
2	96	100
3	112	125

Notice the ***rate of change*** in the linear population shows a *constant* ***addition*** of 16 frogs per year. The ***growth rate*** in the exponential population is less obvious, but is a *constant* ***multiple***, in this case 1.25 or 125% ($\frac{100}{80} = \frac{80}{64} = 1.25$).

Linear equation: y = 16x + 64; where "y" is the population and "x" is the year.
Exponential equation: $y = 64 \bullet (1.25)^x$; where "y" is the population and "x" is the year.

Considering two ordered pairs from a set of data:

(x_1,y_1) & (x_2,y_2)

Slope (rate of change) = $\frac{y_1 - y_2}{x_1 - x_2}$

Growth Rate = $\frac{y_2}{y_1}$

Linear data has:

a *constant* **rate of change**: $\left.\begin{matrix} \frac{80-64}{1-0} \\ \frac{96-80}{2-1} \\ \frac{112-96}{3-2} \end{matrix}\right\}$ = 16 frogs per year

a *variable* **growth rate**:

$\frac{80}{64} = 1.25$ (25% increase)

$\frac{96}{80} = 1.2$ (20% increase)

$\frac{112}{96} = 1.17$ (17% increase)

Exponential data has:

a *variable* **rate of change**:

$\frac{80-64}{1-0}$ = 16 frogs per year

$\frac{100-80}{2-1}$ = 20 frogs per year

$\frac{125-100}{3-2}$ = 25 frogs per year

a *constant* **growth rate**: $\left.\begin{matrix} \frac{80}{64} \\ \frac{100}{80} \\ \frac{125}{100} \end{matrix}\right\}$ = 1.25 (25% increase)

Mathematicians refer to numbers with a constant rate of change as an **arithmetic sequence** and numbers with a constant rate of growth as a **geometric sequence**.

It is easy to find a model by hand for anything that grows or declines by a constant percentage:

Example 4.1.3: Modeling the Value of a Car

A car enthusiast buys two cars:

A 1966 Mustang Fastback for \$23,500, which he believes will increase in value 4%/year.

A 2014 Subaru Outback for \$28,600, which should decrease in value 9%/year.

Find a model for the value of each car over time and determine their expected values in 6 years.

Solution:

Models: let y = value of the car (in dollars), x = time (in years):

Mustang: $y = 23{,}500(1 + .04)^x$ or $y = 23{,}500(1.04)^x$

Subaru: $y = 28{,}600(1 - .09)^x$ or $y = 28{,}600(.91)^x$

Values: Mustang: $y = 23{,}500(1.04)^6 \approx \$29{,}735$

Subaru: $y = 28{,}600(.91)^6 \approx \$16{,}241$

Note: The percentage changes are always relative to 100% and must be subtracted from 1 for a decreasing value and added to 1 for an increasing value.

Example 4.1.4: Modeling Inflation

The price of a gallon of milk is often considered as a model for inflation.

Find a model for the price of a gallon of milk over time if it costs \$4.32 in 2013 and \$4.46 in 2014; and predict the cost in 2020.

Solution:

Model: rate of growth = $\frac{4.46}{4.32} \approx 1.032$ (meaning growth of about 3.2% per year); where y = price of a gallon of milk (in dollars and x = time (in years)

$y = 4.32(1.032)^x$

Value: 2020 would be year 7 since we counted 2013 as the year of the initial value.

$y = 4.32(1.032)^7$ replace x with 7

$y = 4.32(1.2467)$ evaluating $(1.032)^7$

$y \approx 5.39$

Final Answer: If the pattern continues, a gallon on milk will cost \$5.39 in 2020.

Section 4.1: Problem Set

1. The town of Mill Creek has been steadily growing in population.

Year	Population
2004	34670
2005	36480
2006	38390
2007	40280
2008	42620
2009	44570
2010	47220

   a) Make a graph of the data large enough to include a population of 30,000 and the year 2012 (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
   b) Find the growth rate between 2004 and 2005, rounded to 3 decimal places. Explain the meaning of the growth rate in context.
   c) Find the growth rate between 2007 and 2008, rounded to 3 decimal places.
   d) Find the slope between 2004 and 2005, rounded to the nearest whole number. Explain the meaning of the slope in context.
   e) Find the slope between 2007 and 2008, rounded to the nearest whole number.
   f) Add a trend line and estimate the population in 2012.
   g) Use your trend line to estimate the year the population was 30,000.

2. The Southern White Rhino, considered extinct in the 1800's, has enjoyed a steady increase in population in the last hundred years.

Year	Population
1929	150
1948	550
1968	1800
1987	4665
1997	8440
2002	11640
2010	20170

   a) Make a graph of the data large enough to include a population of 35,000 and the year 2015 (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
   b) Find the slope between 1929 and 1948, rounded to the nearest whole number. Explain the meaning of the slope in context.
   c) Find the slope between 1968 and 2002. Explain the meaning of the slope in context.
   d) Find the growth rate between 1929 and 1948, rounded to 3 decimal places. Explain the meaning of the growth rate in context. Hint: the years are not consecutive so you will need to substitute the numbers you know into the equation $y = ab^x$ and solve for b.
   e) Add a trend line and estimate the rhino population in 2015.
   f) Use your trend line to estimate the year the population will reach 35,000.

3. Temperature and pressure are related so that as the vapor pressure is increased the temperature will increase as well. Your refrigerator is designed based on this scientific principle.

Temperature (F^o)	Saturation VP (mbar)
10	2
30	6
45	10
60	18
80	35
90	48
110	88
120	117

   a) Make a graph of the data (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
   b) Find the slope between 10^o and 30^o. Explain the meaning of the slope in context.
   c) Find the growth rate (b) between 10^o and 30^o, rounded to 3 decimal places. Explain the meaning of the growth rate in context. Hint: the temperatures are not consecutive so you will need to substitute the numbers you know into the equation $y = ab^x$ and solve for b.
   d) Add a trend line and estimate the vapor pressure at 70^o.
   e) Use your trend line to estimate the temperature necessary to achieve 100 m-bars of pressure.

4. Temperature and moisture content in wood are related, in that as the temperature increases the percent moisture in the wood decreases.

Temperature(F^o) above ambient	Moisture Content (%)
0	10.0
5	8.5
10	7.3
15	6.3
20	5.5
25	4.8
30	4.2

   a) Make a graph of the data (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
   b) Find the slope between 10^o and 30^o. Explain the meaning of the slope in context.
   c) Find the growth rate (b) between 20^o and 25^o, rounded to 3 decimal places. Explain the meaning of the growth rate in context. Hint: the temperatures are not consecutive so you will need to substitute the numbers you know into the equation $y = ab^x$ and solve for b.
   d) Add a trend line and estimate the moisture content at 35^o.
   e) Use your trend line to estimate the temperature necessary to achieve 6% moisture content.

5. Open source is a growing movement where people create software, apps or curriculum and make it available for free. The chart shows the number of open source software projects considering May of 1999 as month 0.

Month	Projects
0	125
33	750
42	1000
54	1500
66	2450
75	3500

   a) Make a graph of the data large enough to include month 80 (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
   b) Find the slope between months 33 and 42, rounded to 1 decimal place. Explain the meaning of the slope in context.
   c) Find the growth rate (b) between month 33 and 42, rounded to 3 decimal places. Hint: the months are not consecutive so you will need to substitute the numbers you know into the equation $y = ab^x$ and solve for b.
   d) Find the growth rate (b) between month 54 and 66, rounded to 3 decimal places. Hint: the months are not consecutive so you will need to substitute the numbers you know into the equation $y = ab^x$ and solve for b.
   e) Add a trend line and estimate the number of open source projects you would expect in month 80.
   f) Use your trend line to estimate the month there would have been 2000 projects.

6. Children are more likely to be born with Down's syndrome as the age of the mother increases.

Age	Percent DS Births
33	0.2
38	0.5
41	1.2
43	2.1
45	3.6

   a) Make a graph of the data (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
   b) Find the slope between age 41 and 43, rounded to 1 decimal place. Explain the meaning of the slope in context.
   c) Find the growth rate (b) between age 33 and 38, rounded to 3 decimal places. Hint: the months are not consecutive so you will need to substitute the numbers you know into the equation $y = ab^x$ and solve for b.
   d) Find the growth rate (b) between age 41 and 43, rounded to 3 decimal places. Hint: the months are not consecutive so you will need to substitute the numbers you know into the equation $y = ab^x$ and solve for b.
   e) Add a trend line and estimate the percent of Down's syndrome births you would expect from 42 year old mothers.

7. Data concerning the relationship between weight and length in the walleye pike are shown in the table.

Length (cm)	Weight (kg)
20	0.1
30	0.2
40	0.4
50	0.8
60	1.4

a) Make a graph of the data large enough to include 70 centimeters and 4 kilograms (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
b) Find the slope between 40 and 50 centimeters. Explain the meaning of the slope in context.
c) Find the growth rate (b) between 40 and 50 centimeters, rounded to 3 decimal places. Hint: the months are not consecutive so you will need to substitute the numbers you know into the equation $y = ab^x$ and solve for b.
d) Find the growth rate (b) between 50 and 60 centimeters, rounded to 3 decimal places. Hint: the months are not consecutive so you will need to substitute the numbers you know into the equation $y = ab^x$ and solve for b.
e) Add a trend line and estimate the weight you would expect of a 70 cm walleye.
f) Use your trend line to estimate the length a walleye would have to reach to weigh 4 kg.

Section 4.2: Finding Exponential Equations

In this section we learn how to find an exponential equation from ordered pairs using regression.

Again, the equation gives you accurate answers between the data points and predictive power beyond them.

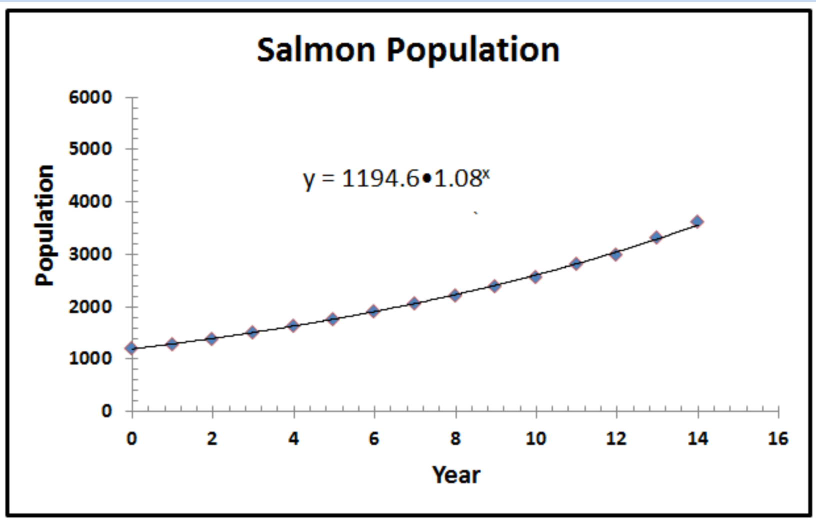
Reconsider the problem introduced at the beginning of the chapter regarding salmon population:

Example 4.2.1: Finding an Exponential Equation

Consider the data in the table and use the regression feature of your graphing calculator to find an exponential equation to model the data. Then use the equation to predict the population in year 25.

Year	Population
0	1200
1	1296
2	1400
3	1512
4	1633
5	1763
6	1904
7	2062
8	2220
9	2405
10	2568
11	2819
12	3008
13	3326
14	3634

Solution:
Enter the 15 ordered pairs into your calculator and choose ExpReg. See if you agree with the equation listed on the graph.



$y = 1194.6(1.08)^x$

$y = 1194.6(1.08)^{25}$ replace x with 25

$y = 1194.6(6.848475)$ exponents first

$y = 8181$ multiplication last

Final Answer: If the pattern continues, the salmon population will be approximately 8181 in year 25.

Note: Recall from the introduction to chapter 4 that exponential equations take the form $y = ab^x$; where "a" represents the initial population, "b" represents the rate of growth, and "x" is the time. We can come up with an equation by hand letting a = 1200 and b = $\frac{3326}{3008}$ = 1.106. It is guess work, though, to decide which two years to use for the growth rate.
The model $y = 1200 \cdot 1.106^x$ (from the example at the beginning of the chapter) predicts a population of 14,096 salmon at year 25.
Regression considers all 15 ordered pairs, finding an "average" growth rate, and thus comes up with a more accurate exponential model.

Consider a practical problem where the data is decreasing:

Example 4.2.2: Finding an Exponential Equation

Consider the exponential river level data provided in the table:

a) Find an exponential equation to model the data by hand
b) Use the regression feature of your graphing calculator to find an exponential equation to model the data.

Day	Level (CFS)
0	76
1	64
2	53
3	45
4	38
5	32
6	27
7	23
8	20
9	18

Solution:

a) Recall that exponential equations take the form $y = ab^x$; where "a" represents the initial population, "b" represents the growth rate (or decline rate), and "x" is the time.
The initial river level (a) is 76.
Exponential equations have a constant growth (decline) rate. Picking a few random points we see a fairly constant decline rate.

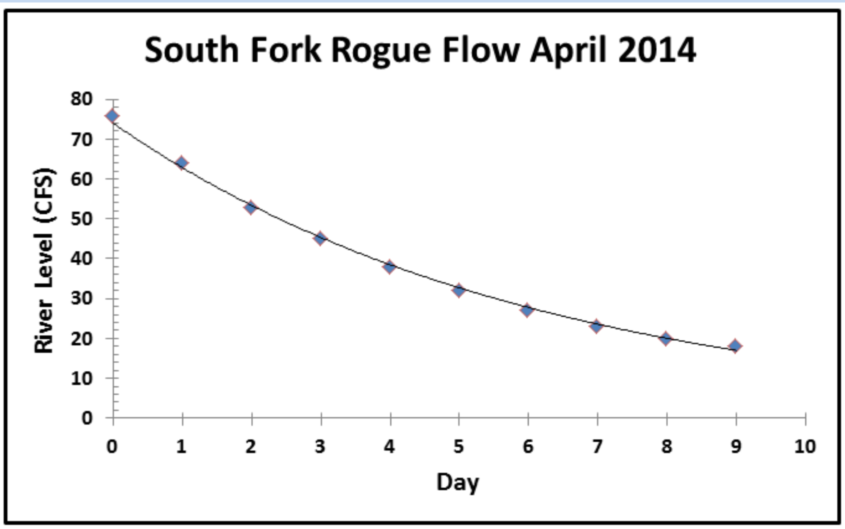
$\frac{64}{76} \approx 0.842$ $\quad$ $\frac{53}{64} \approx 0.828$ $\quad$ $\frac{45}{53} \approx 0.849$

… which average to about 0.843

$\frac{32}{38} \approx 0.842$ $\quad$ $\frac{27}{32} \approx 0.844$ $\quad$ $\frac{23}{27} \approx 0.852$

The exponential model would be: $y = 76 \bullet 0.843^x$

b) Entering the 10 ordered pairs into the STAT button in the calculator and choosing ExpReg yields the equation:
$y = 74.04 \bullet 0.849^x$



Note: The rate of approximately 85% (0.849 ≈ .85 or 85%) can be interpreted as a daily decrease of 15%. Regression is obviously preferable to finding the equation by hand, both as a time-saver and in terms of accuracy; but it is a confidence building exercise to rely on your own knowledge.

Section 4.2: Problem Set

1. The town of Mill Creek's population growth is shown the chart.

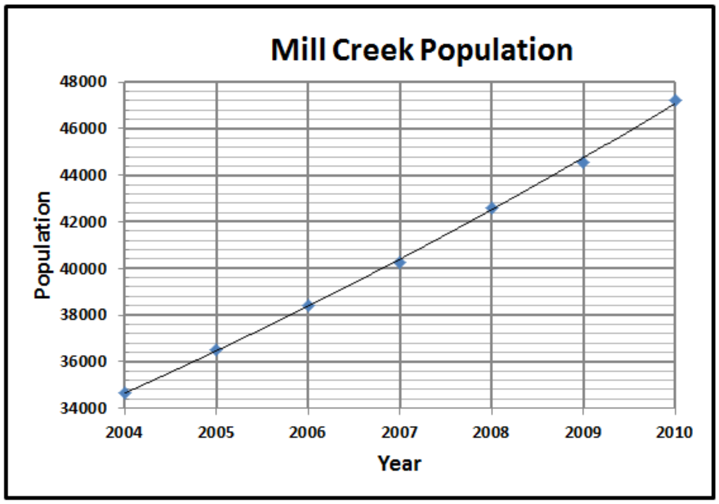
Year	Population
2004	34670
2005	36480
2006	38390
2007	40280
2008	42620
2009	44570
2010	47220

   a) Consider 2004 to be year 0 and find an exponential equation to model the data by hand. Round the numbers in your equation to 3 decimal places.

   b) Consider 2004 to be year 0 and use regression to find an exponential equation to model the data. Round the numbers in your equation to 3 decimal places.

   c) Use your regression equation to predict the population in 2018. Round to the nearest person.

   d) Use your regression equation to estimate the population in 1998. Round to the nearest person.



2. The Southern White Rhino, considered extinct in the 1800's, has enjoyed a steady increase in population in the last hundred years.

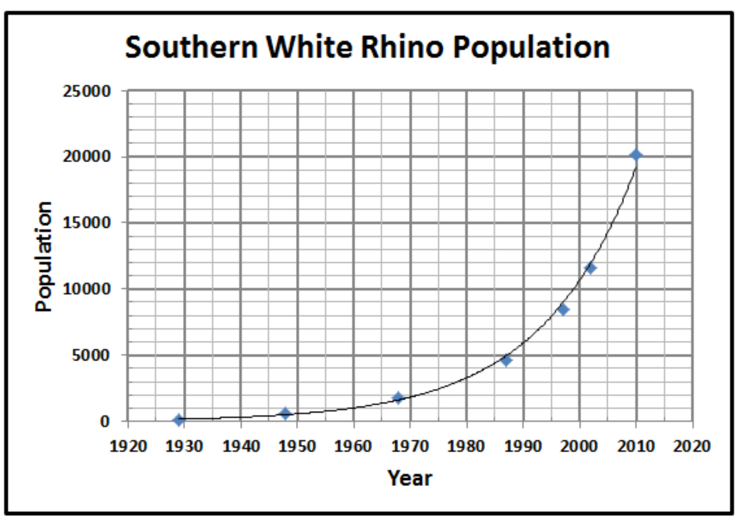
Year	Population
1929	150
1948	550
1968	1800
1987	4665
1997	8440
2002	11640
2010	20170

   a) Consider 1929 to be year 0 and use regression to find an exponential equation to model the data. Round the numbers in your equation to 3 decimal places.

   b) Use your equation to estimate the population in 2006. Round to the nearest animal.

   c) Use your equation to predict the population in 1920. Round to the nearest animal.

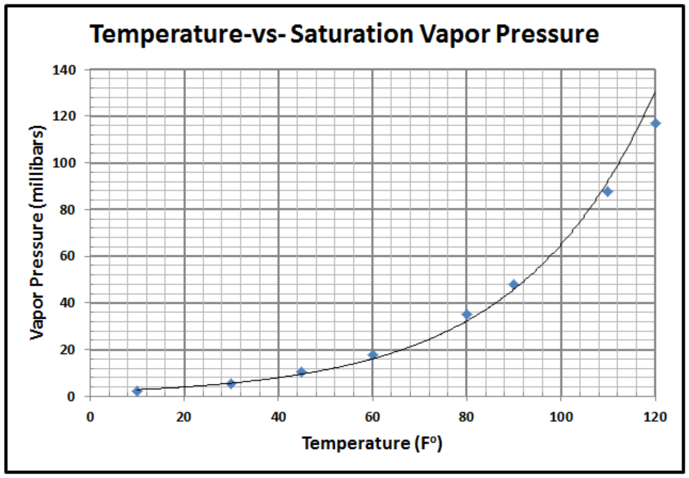
   d) Find the growth rate and explain its meaning in context of the data.



3. Temperature and pressure are related so that as the vapor pressure is increased the temperature will increase as well. Your refrigerator is designed based this scientific principle.

Temperature (F)	Saturation VP (mbar)
10	2
30	6
45	10
60	18
80	35
90	48
110	88
120	117

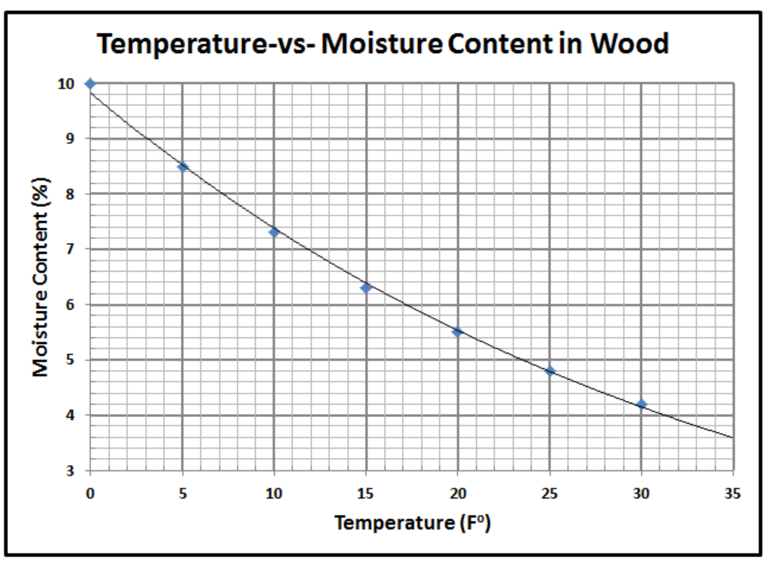
   a) Use regression to find an exponential equation to model the data. Round the numbers in your equation to 3 decimal places.
   b) Use your equation to accurately find the vapor pressure for a temperature of 100°, accurate to 1 decimal place.
   c) Use your equation to predict the vapor pressure for a temperature of 130°, accurate to 1 decimal place.



4. Temperature and moisture content in wood are related, in that as the temperature increases the percent moisture in the wood decreases.

Temperature (F above ambient)	Moisture Content (%)
0	10.0
5	8.5
10	7.3
15	6.3
20	5.5
25	4.8
30	4.2

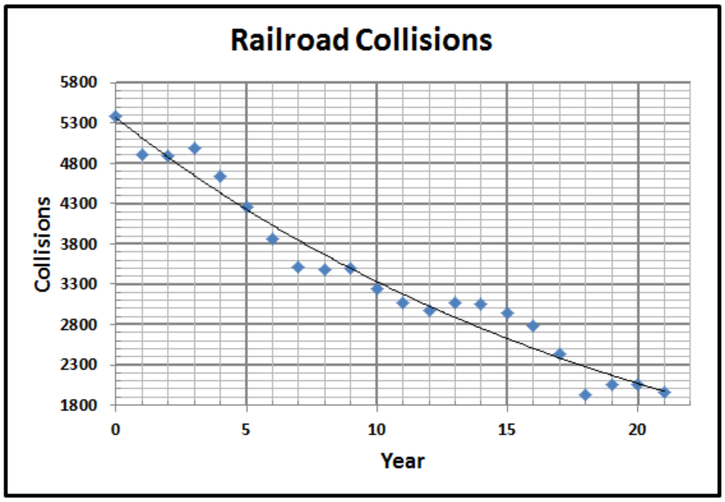
   a) Use regression to find an exponential equation to model the data. Round the numbers in your equation to 3 decimal places.
   b) Use your equation to accurately find the moisture content a temperature of 35°, accurate to 1 decimal place.
   c) Use your equation to accurately find the moisture content a temperature of 3°, accurate to 1 decimal place.
   d) Find the growth (decline) rate and explain its meaning in context of the data.



5. Reconsider the data regarding collisions between trains and automobiles that we modeled with a power equation in chapter 3.

Year	Collisions
1990	5388
1991	4910
1992	4892
1993	4979
1994	4633
1995	4257
1996	3865
1997	3508
1998	3489
1999	3502
2000	3237
2001	3077
2002	2977
2003	3077
2004	3057
2005	2936
2006	2776
2007	2429
2008	1934
2009	2052
2010	2062
2011	1960

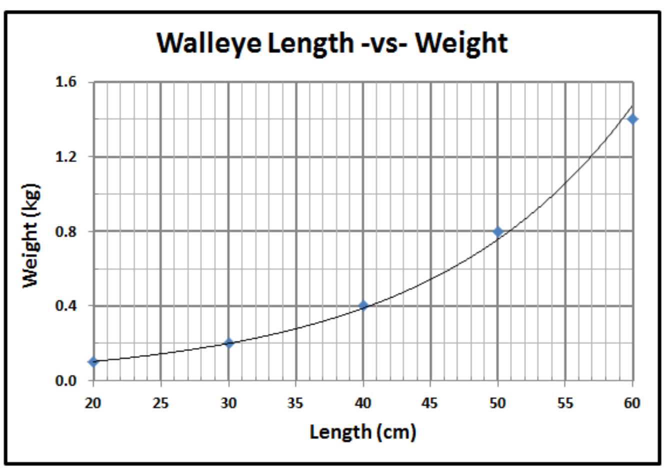
a) Consider 1990 to be year 0 and use regression to find an exponential equation to model the data. Round the numbers in your equation to 3 decimal places.
b) Use your equation to predict the number of collisions in 2000 (compare to the actual data). Round to the nearest collision.
c) Use your equation to estimate the number of collisions in 1980 (before the data began being collected). Round to the nearest collision.
d) Use your equation to predict the number of collisions in 2018. Round to the nearest collision.
e) Find the growth rate and explain its meaning in context of the data.



6. Length and weight are naturally related in animals. Consider the data for the walleye pike in the table.

Length (cm)	Weight (kg)
20	0.1
30	0.2
40	0.4
50	0.8
60	1.4

a) Use regression to find an exponential equation to model the data. Round the numbers in your equation to 3 decimal places.
b) Use your equation to accurately predict the weight of a 70 centimeter walleye, accurate to the nearest tenth.
c) Use your equation to accurately predict the weight of an 80 centimeter walleye, accurate to the nearest tenth.



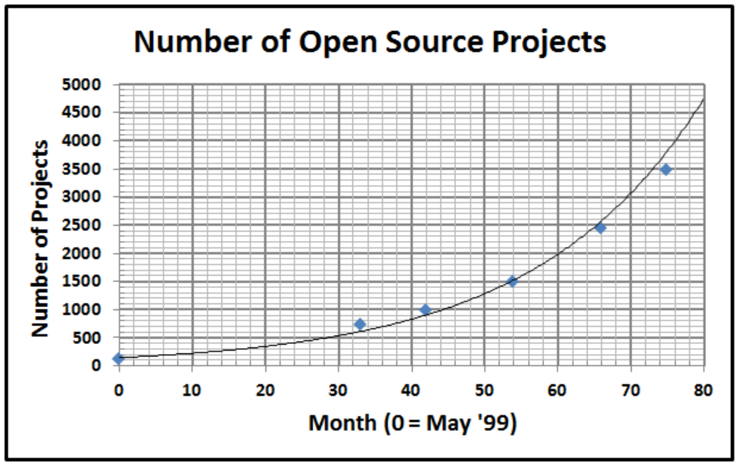
7. Open source is a growing movement where people create software, apps or even curriculum and make it available for free. The chart shows the number of open source software projects considering May of 1999 as month 0.

Month	Projects
0	125
33	750
42	1000
54	1500
66	2450
75	3500

a) Use regression to find an exponential equation to model the data. Round the numbers in your equation to 3 decimal places.

b) Use your equation to accurately predict the number of open source projects in month 96 (year 8). Round to the nearest project.

c) Use your equation to accurately predict the number of open source projects in month 120 (year 10). Round to the nearest project.



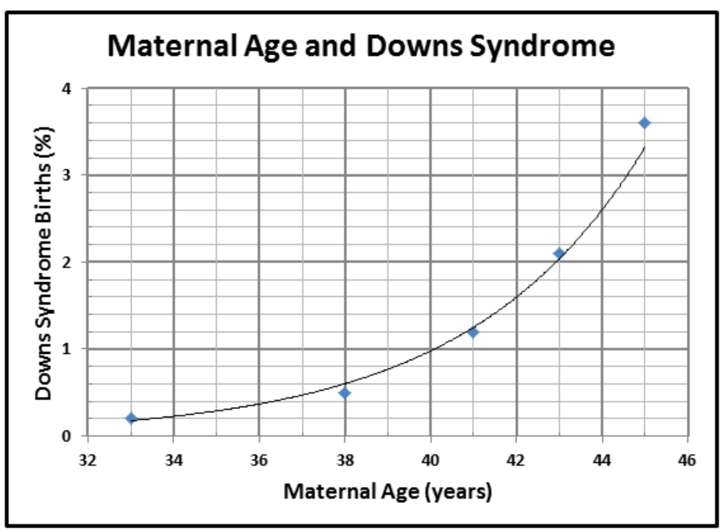
8. Children are more likely to be born with Down's syndrome as the age of the mother increases.

Age	Percent DS Births
33	0.2
38	0.5
41	1.2
43	2.1
45	3.6

a) Use regression to find an exponential equation to model the data. Round the numbers in your equation to 3 significant digits (i.e. non-zero values). Note: The "a" value in the regression equation is very small and is listed in scientific notation.

b) Use your equation to accurately estimate the percent of Down's syndrome births that can be expected from 44 year old mothers, rounded to 1 decimal place.

c) Use your equation to accurately predict the percent of Down's syndrome births that can be expected from 46 year old mothers, rounded to the tenth place.



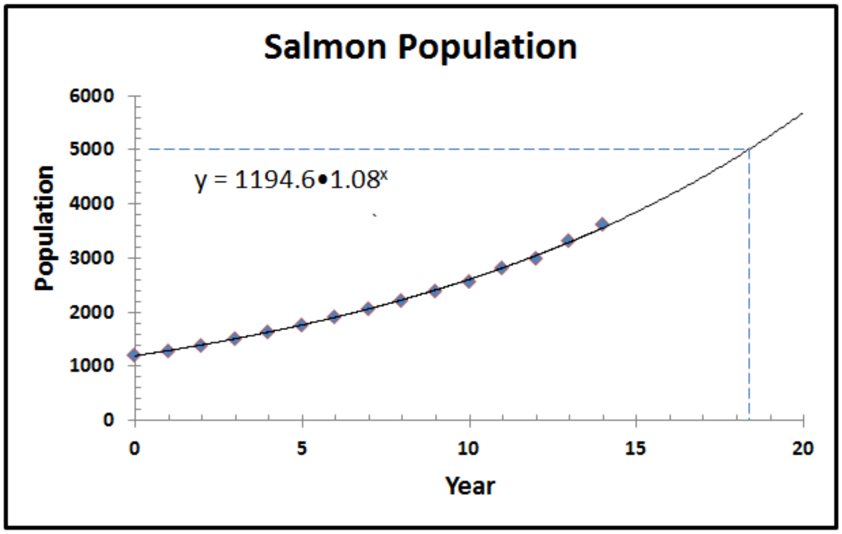
Section 4.3: Using Exponential Equations

We now turn our attention to the complex and practical skill of finding an input "x" that will produce an output "y" that we are interested in.

Suppose a biologist wants to know what year the salmon population will reach 5000.

We can extend the trend line to give us an approximate answer.

We could simply guess values for "x" until we get it close enough (you may actually decide this is the best way since solving an exponential equation requires considerable skill).



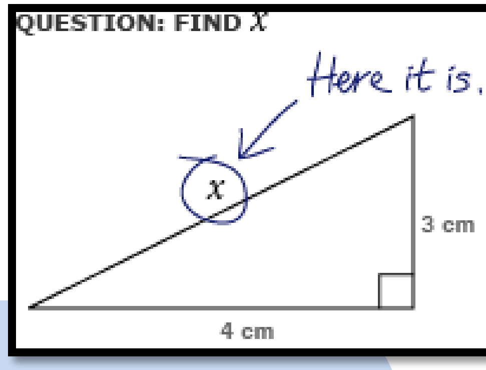
$1194.6(1.08)^{16}$	guess 16 … y ≈ 4093 … too low!
$1194.6(1.08)^{24}$	guess 24 … y ≈ 7575 … too high!
$1194.6(1.08)^{20}$	guess 20 … y ≈ 5568 … too high!
$1194.6(1.08)^{18}$	guess 18 … y ≈ 4774 … too low!
$1194.6(1.08)^{19}$	guess 19 … y ≈ 5156 … too high!
$1194.6(1.08)^{18.5}$	guess 18.5 … y ≈ 4961 … close!
$1194.6(1.08)^{18.6}$	guess 18.6 … y ≈ 4999 … close enough!

Sometime near the middle of year 18 then the population will reach 5000 salmon.

It is important to recognize that this may be the most practical use for the equation. A biologist might want to plan ahead for a river to be reopened to fishermen based on fish counts.

Guessing is a slow process and may not produce an accurate enough answer for some applications; more importantly though … guessing just isn't cool when there is a better way.

How can we solve for "x" without guessing? $5000 = 1194.6(1.08)^x$



Solving, simply means finding "x". We need to get rid of the 1194.6 and the 1.08 … to find "x".

When we knew "x", we used the order of operations (PEMDAS) to simplify. Now we need to free "x", so we remove the numbers in the reverse order of operations (SADMEP) using the inverse operation.

$y = 1194.6 \bullet 1.08^x$

$5000 = 1194.6 \bullet 1.08^x$

$4.1855 \approx 1.08^x$ remove the multiplication first by dividing both sides by 1194.6

Now we need some help. We used the fact that the inverse of multiplying is dividing when removing the 1194.6; but how can we bring the x down from its position as an exponent?

The **logarithm** was invented for this purpose; much as the square root was invented as the inverse for squaring … I will explain more after this example.

$x \approx \text{Log}_{1.08}4.1855$	introduce a logarithm as the inverse of an exponent (read as Log of 4.1855 base 1.08)
$x \approx \text{Log}(4.1855)/\text{Log}(1.08)$	how it is entered in the calculator
$x \approx 18.602$	calculator result

We get a number just above 18.6, as we expected from our earlier guessing.

So what is a logarithm?

If you are solving the equation $x^2 = 48$, you have accepted that you take the square root of both sides to get rid of the exponent 2, thus solving for x. Square and square root are inverse operations. Consider how you would explain what $\sqrt{48}$ means to someone unfamiliar with square roots. You inevitably explain that $\sqrt{48}$ produces the number that when squared is 48. You should even be able to estimate the calculator's answer as just under 7, since $7^2 = 49$.

Notice that you **<u>cannot</u>** directly explain what a square root is; you are forced to explain it as the inverse of squaring. What the calculator does to find a square root is an interesting topic for another time.

In the same way, a logarithm can only be understood when explained as an inverse of an exponent. $\text{Log}_2 8$ (log of 8 base 2) tells you what power 2 must be raised to equal 8. In other words, it solves the equation $2^x = 8$. Notice that, like the $\sqrt{25}$, the solution for "x" is a whole number and can be known without a calculator since $2^3 = 8$. Enter Log(8)/Log(2) to see that the answer is 3. Notice also that 2 is the **base** of the exponent and is also the **base** in the logarithm (lower number in $\text{Log}_2 8$).

Can you estimate $\text{Log}_4 20$? 4 to what power is 20? 4^2 is 16 and 4^3 is 64.

So a good estimate for the $\text{Log}_4 20$ might be 2.1. Notice the calculator gives the more precise answer, $\text{Log}(20)/\text{Log}(4) \approx 2.160964$ … this is called an **irrational number** and must be rounded as it is a decimal that never ends and does not have any discernable pattern.

What the logarithm does can be easily understood. How a calculator finds it requires an amazing discovery of an infinite series, which is not encountered until a deeper algebra course. Understanding where this infinite series comes from involves calculus. Brings to mind Alice in Wonderland's rabbit hole; this is what makes math so beautiful and terrifying. Step in and you will find more adventure and wonder the deeper you go.

Do not worry if this more terrifying than beautiful for you; in this course we will only brace ourselves safely at the entrance and look in.

We best do another practical example:

Example 4.3.1: Solving an Exponential Equation

Suppose we want to model the decrease in a river's flow in the summer to predict when it will be 300 CFS. We observe the level to be 1860 CFS (cubic feet per second) on Monday, 1702 CFS on Tuesday, and 1557 CFS on Wednesday.
Recall from section 4.1 that exponential equations take the form $y = ab^x$; where "a" represents the initial population, "b" represents the rate of growth, and "x" is the time.

Solution:

Notice that $\frac{1702}{1860} \approx 0.915$ and $\frac{1557}{1702} \approx 0.915$ so you could say the level is dropping around 8.5% per day, or each day you have about 91.5% of what you had the day before.
The exponential equation $F = 1860 \bullet 0.915^d$ is a good model since the level is dropping at a consistent rate; where F = flow and d = day.
We want to know what day the flow will drop to 300 CFS.

$F = 1860 \bullet 0.915^d$

$300 = 1860 \bullet 0.915^d$ replace F with 300

$.1613 \approx 0.915^d$ divide both sides by 1860

$d \approx \text{Log}_{0.915} 0.1613$ introduce logarithms to solve for d

$d \approx 20.5$ enter Log(0.1613)/Log(0.915) in your calculator

Final Answer: If the pattern continues, the level will hit 300 CFS about half way through day 20. This would be on Sunday about 3 weeks away.

Note: It is important to recognize that growth or decline is always measured relative to 100%. If a population of 490 elk is increasing at 13% the model would be $y = 490(100\%+13\%)^x$ or $y = 490 \bullet 1.13^x$. If the population was decreasing at 13% the model would be $y = 490(100\%-13\%)^x$ or $y = 490 \bullet 0.87^x$.

We could have also entered our data into the calculator and found our model using exponential regression. Remember that regression is better when the percent decrease is less consistent since regression will take an average.

0	1860
1	1702
2	1557

It is tempting to assume that any upward tending curve will be best modeled by an exponential equation. This shape could also be modeled, however by the right half of the parabola that we studied in chapter 2.

Notice that the researcher modeling the relationship between the height of a tree and the volume of growth chose to do just that. In chapter 6 we will work on choosing the best model.

Figure 2 to determine maximum potential annual volume growth increment in cm3 per year for a tree of a given height. The quadratic equation was the best fit for the maximum site potential tree data (R2= 0.96, p<0.0001) until tree heights approached 1.4 meters, below which it underestimated maximum potential growth.

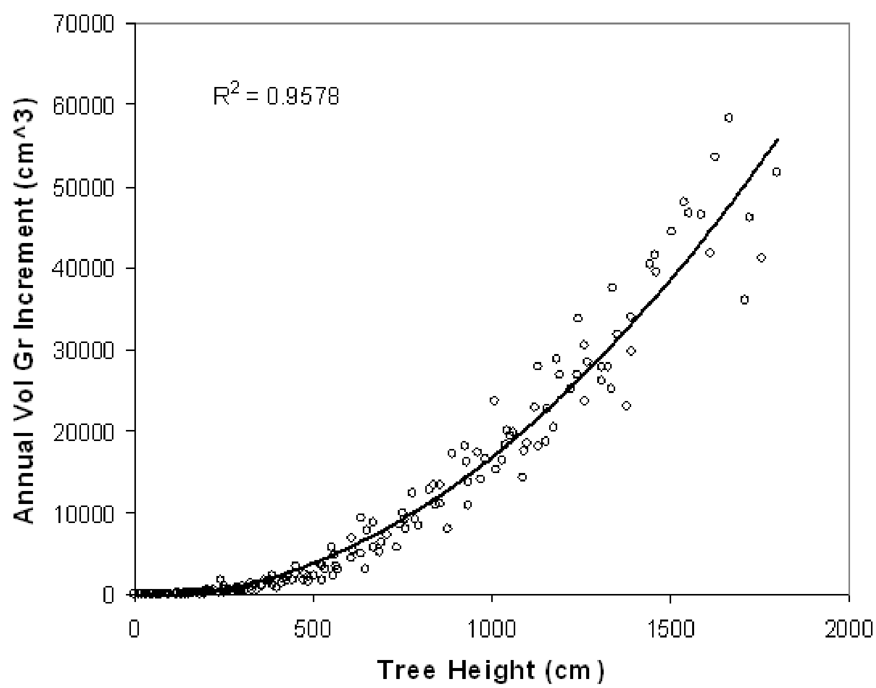


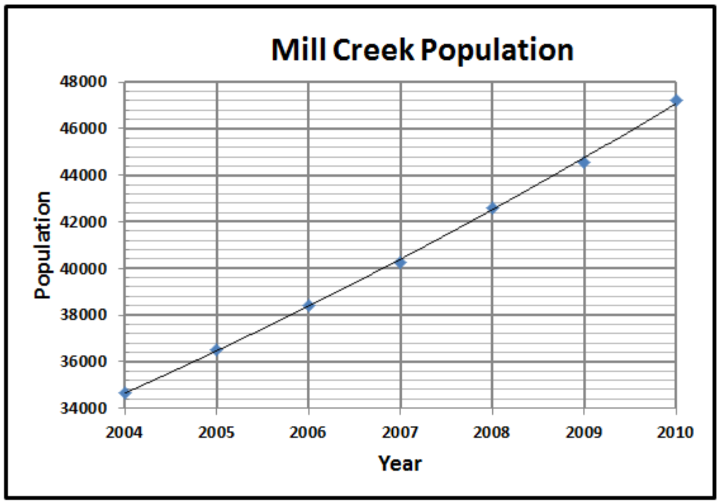
Figure 2: Scatterplot and fitted regression line for maximum site potential height (Ht) to volume growth curve. *Each data point represents the annual volume growth increment for a single height of a tree. Multiple data points for each of the 19 trees are included y = 0 .018x2 - 0.27x - 466.74 (R2= 0.96, p<0.0001)*

Section 4.3: Problem Set

1. The town of Mill Creek's population growth is shown the chart.

Year	Population
2004	34670
2005	36480
2006	38390
2007	40280
2008	42620
2009	44570
2010	47220

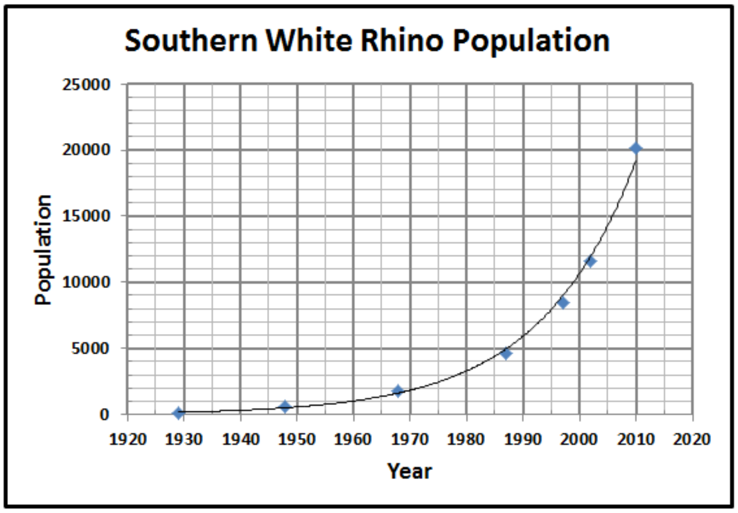
   a) Consider 2004 to be year 0 and use regression to find an exponential equation to model the data. Round the numbers in your equation to 3 decimal places.
   b) Use your regression equation to find the year the population will reach 60,000, accurate to 1 decimal place.
   c) Use your regression equation to find the year the population was 25,000, accurate to 1 decimal place.



2. The Southern White Rhino, considered extinct in the 1800's, has enjoyed a steady increase in population in the last hundred years.

Year	Population
1929	150
1948	550
1968	1800
1987	4665
1997	8440
2002	11640
2010	20170

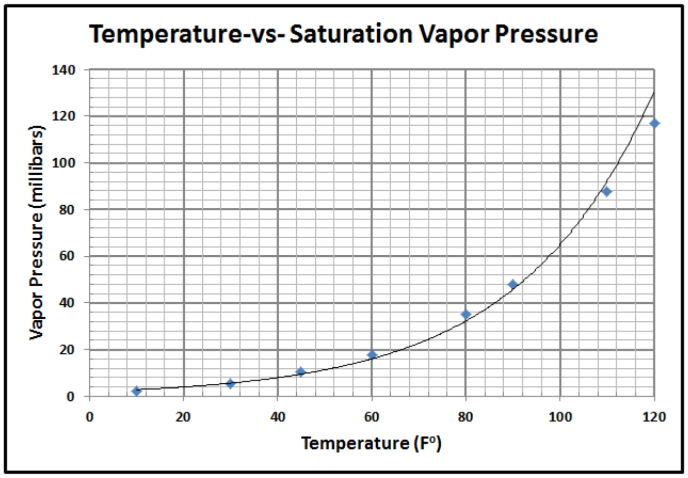
   a) Consider 1929 to be year 0 and use regression to find an exponential equation to model the data. Round the numbers in your equation to 3 decimal places.
   b) It is clear in the table that the population reached 1000 between 1948 and 1968. Use your equation to find the year, accurate to the nearest tenth place.
   c) Use your equation to predict the year the population will reach 30,000, accurate to the nearest tenth place.



3. Temperature and pressure are related so that as the vapor pressure is increased the temperature will increase as well. Your refrigerator is designed based this scientific principle.

Temperature (F)	Saturation VP (mbar)
10	2
30	6
45	10
60	18
80	35
90	48
110	88
120	117

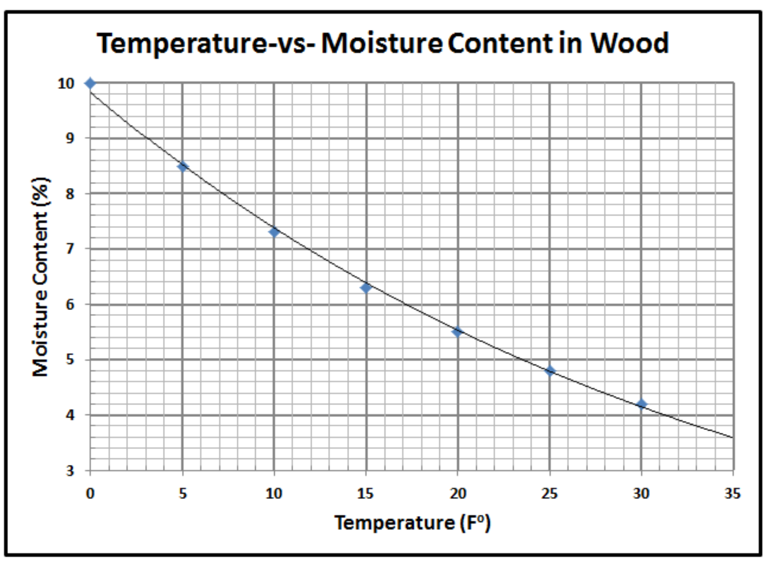
   a) Use regression to find an exponential equation to model the data. Round the numbers in your equation to 3 decimal places.
   b) It is clear in the table that the vapor pressure reached 100 m-bars between 110° and 120°. Use your equation to accurately find the temperature accurate to the nearest tenth place.
   c) Use your equation to predict the temperature the vapor will reach at 140 m-bars of pressure accurate to the nearest tenth place.



4. Temperature and moisture content in wood are related, in that as the temperature increases the percent moisture in the wood decreases.

Temperature (F above ambient)	Moisture Content (%)
0	10.0
5	8.5
10	7.3
15	6.3
20	5.5
25	4.8
30	4.2

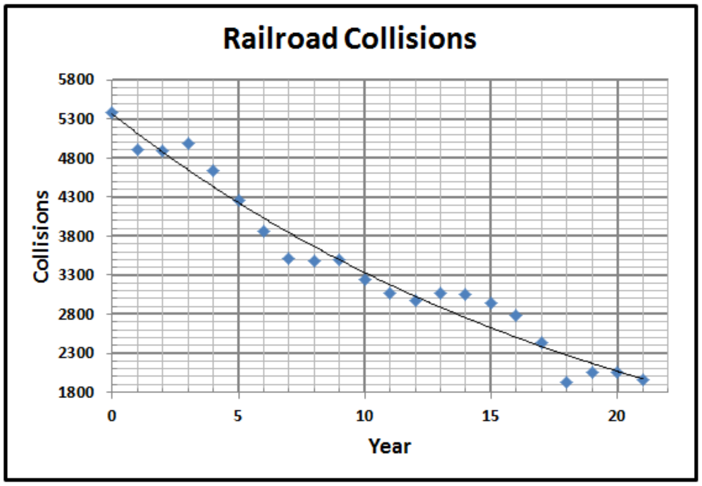
   a) Use regression to find an exponential equation to model the data. Round the numbers in your equation to 3 decimal places.
   b) Use your equation to accurately find the temperature, accurate to the nearest tenth place, needed to get the moisture content to 5%.
   c) Use your equation to predict the temperature, accurate to the nearest tenth place that would be necessary to get the moisture content to 2%.



5. Reconsider the data regarding collisions between trains and automobiles that we modeled with a power equation in chapter 3.

   a) Consider 1990 to be year 0 and use regression to find an exponential equation to model the data. Round the numbers in your equation to 3 decimal places.
   b) Use your equation to predict the year that the number of collisions will drop to 1800, accurate to 1 decimal place.
   c) Use your equation to predict the year that the number of collisions will drop to 1650, accurate to 1 decimal place.

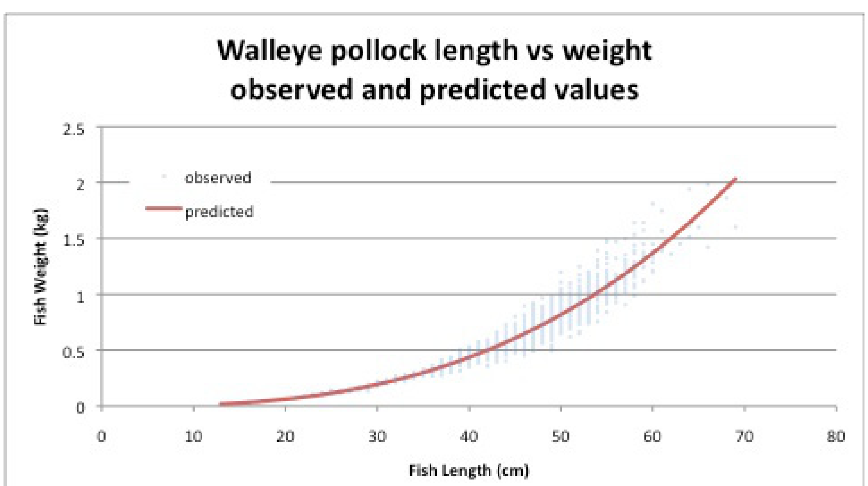
Year	Collisions
1990	5388
1991	4910
1992	4892
1993	4979
1994	4633
1995	4257
1996	3865
1997	3508
1998	3489
1999	3502
2000	3237
2001	3077
2002	2977
2003	3077
2004	3057
2005	2936
2006	2776
2007	2429
2008	1934
2009	2052
2010	2062
2011	1960



6. The walleye (a fish) naturally grows in length and width simultaneously.

Length (cm)	Weight (kg)
20	0.1
30	0.2
40	0.4
50	0.8
60	1.4

   a) Use regression to find an exponential equation to model the data. Round the numbers in your equation to 3 decimal places.
   b) Use your equation to find the length a walleye would need to be to weigh 3 kg. Answer to the nearest centimeter.
   c) Use your equation to find the length a walleye would need to be to weigh 5 kg. Answer to the nearest centimeter.



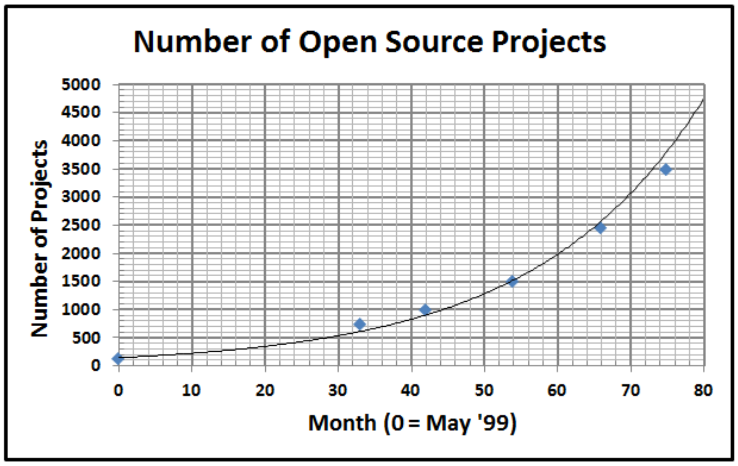
7. The chart shows the number of open source software projects considering May of 1999 as month 0.

Month	Projects
0	125
33	750
42	1000
54	1500
66	2450
75	3500

a) Use regression to find an exponential equation to model the data. Round the numbers in your equation to 3 decimal places.

b) Use your equation to accurately find the month when the number of projects reached 2000, accurate to 1 decimal place.

c) Use your equation to predict the month when there will be 10,000 open source projects, accurate to 1 decimal place.



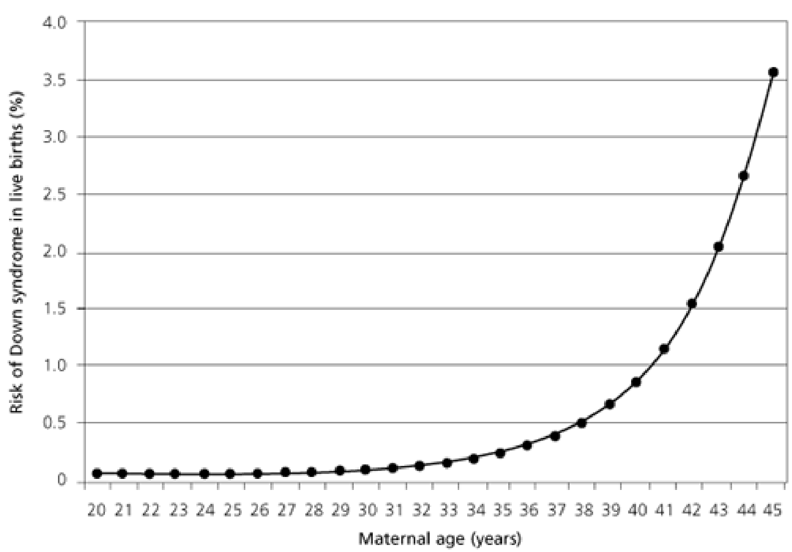
8. The percent of downs syndrome births increase with the age of the mother.

Age	Percent DS Births
33	0.2
38	0.5
41	1.2
43	2.1
45	3.6

a) Use regression to find an exponential equation to model the data. Round the numbers in your equation to 3 significant digits (i.e. non-zero values). Note: The "a" value in the regression equation is very small and is listed in scientific notation.

b) Use your equation to find the mother's age that would have a 5% risk for downs syndrome.

c) Use your equation to find the mother's age that would have a 10% risk for downs syndrome.



Chapter 5:

Logarithmic Relationships

Douglas fir

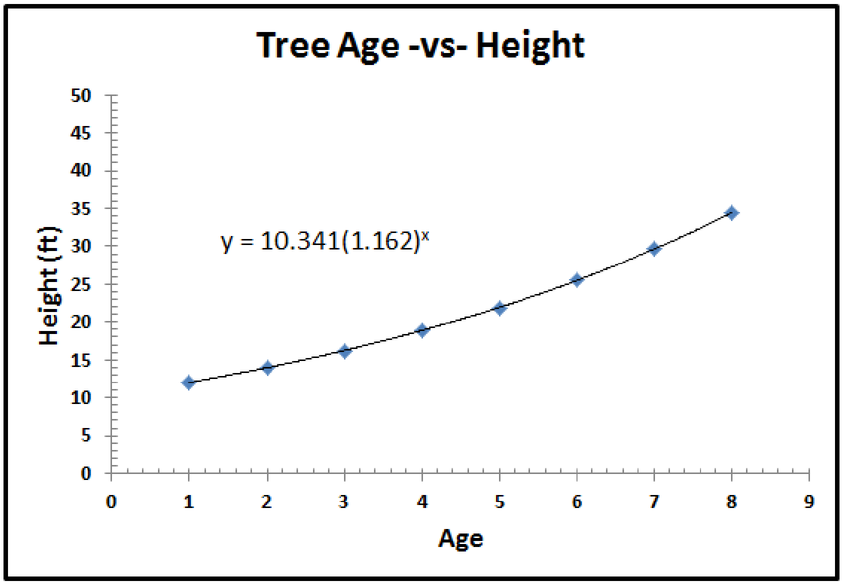
As we saw in the chapter 4 exponential and logarithmic equations are related.

Foresters use logarithmic equations to model the relationship between the height and age of a tree.

We saw in chapter 4 that a logarithm had to be introduced when solving for x in an exponential equation.

Consider the tree data to the right and its exponential model $y = 10.341(1.162)^x$.

Year	Height (ft)
1	12.0
2	14.0
3	16.2
4	18.9
5	21.9
6	25.5
7	29.6
8	34.4

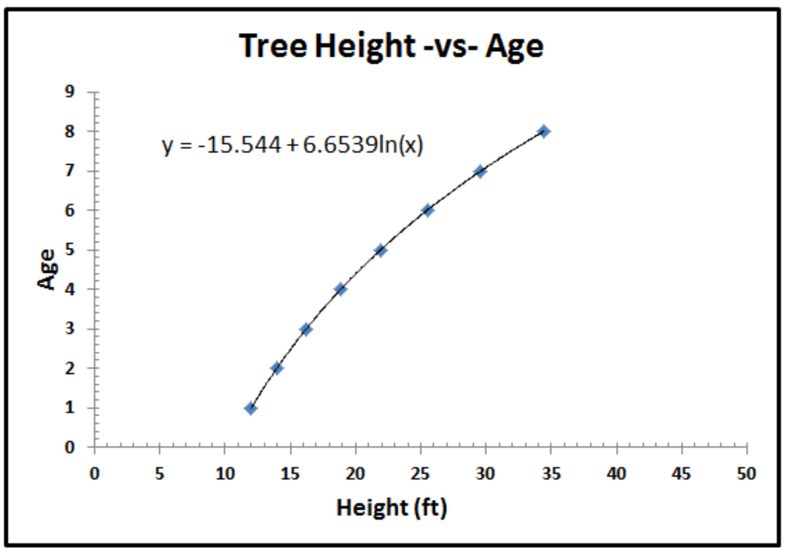


This exponential equation is designed to take age as an input and give the height as an output.

We saw in section 4.3 that the equation can be used, often more practically than the graph, to predict the age for a given height. If a biologist wanted to know when the tree could be predicted to reach 50 feet tall s/he could plug y = 50 into the equation and use logarithms to find $x \approx 10.5$ years.

Height (ft)	Year
12.0	1
14.0	2
16.2	3
18.9	4
21.9	5
25.5	6
29.6	7
34.4	8

If the biologist's goal in modeling the data was to predict the age for a given height it is more straight forward to just consider the height as the independent variable and find a logarithmic model.



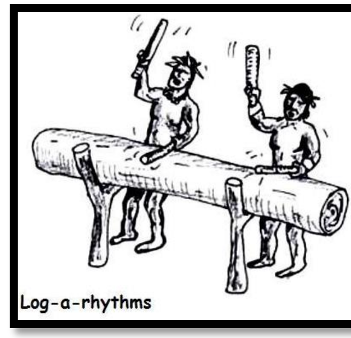
Now enter: $-15.544 + 6.6539\ln(50)$ and get the same age ≈ 10.5 years.

Notice the calculator employs the natural logarithm (ln), which is right below the log button on your calculator.

It turns out that there are two types of logarithms, and we will have to stick our heads a little further into the rabbit hole to make sense of them. Logarithmic and natural logarithmic equations will be our point of discussion in chapter 5.

Section 5.1: The Shape of a Logarithmic Equation

John Napier published his invention of the logarithm in 1614; pictured here in a stylish ruff and trendy beard-goatee.



We did not need to know this before, but logarithms are generically base 10. If you enter log(90) into your calculator it returns approximately 1.954. This is because of the *hidden* base 10. Log(90) is understood to be $\log_{10} 90$; remember this answers the question 10 to what power is 90 ($10^x = 90$)? Since $10^2 = 100$, it makes sense that the calculator gives a result just below 2.

Let's look at a generic logarithmic equation and find its graph:

Example 5.1.1: Graphing a Logarithmic Equation

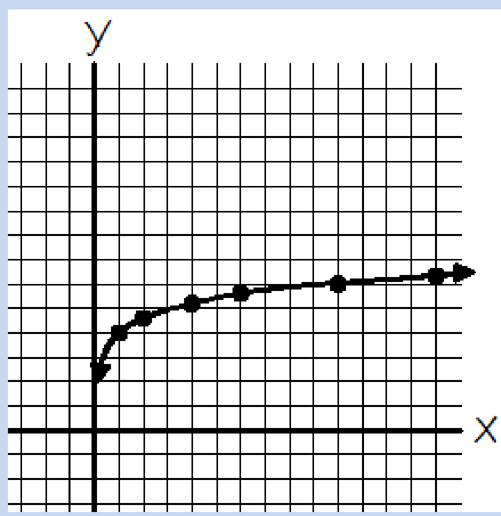
Find some ordered pairs then graph the equation: $y = 4 + 2\log x$

Solution:

You are free to pick any positive value for "x" in the logarithmic equation; negative numbers do not work since there is no power you can raise 10 to that will result in a negative number.

A partial list of ordered pairs is shown in the table.

x	y
1	4.0
2	4.6
4	5.2
6	5.6
10	6.0
14	6.3



$y = 4 + 2\log x$

$y = 4 + 2\log(6)$ replace x with 6

$y \approx 4 + 2(.7782)$ enter log(6) into your calculator

$y \approx 4 + 1.5564$ multiplication first

$y \approx 5.6$ addition next

Note: Notice that logarithmic graphs increase quickly at first then level off over time while the exponential graphs of chapter 4 increased slowly at first then dramatically over time. Logarithmic equations do a better job of modeling things that level off in their growth like alligators, trees or populations with limited resources (i.e. fish in a pond).

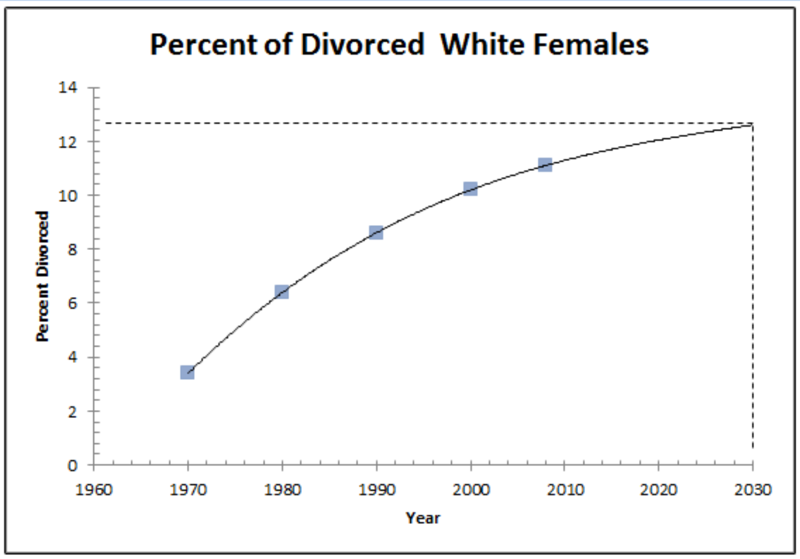
Consider the divorce data for white females over the age of 15 in the chart:

Example 5.1.2: Graphing Logarithmic Data

Graph the data and make a prediction for the divorce rate in 2030.

Year	Percent Divorced
1970	3.4
1980	6.4
1990	8.6
2000	10.2
2008	11.1

Solution:



The graph shows a trend that levels off over time just like the shape of the graph we saw in example 5.1.1.

A good estimate for the divorce rate would be just under 13%, based on the trend.

Note: Notice the rate of change (slope) can be seen to be declining even before graphing.

$$\frac{6.4-3.4}{1980-1970} = 0.3, \textit{meaning an increase of } 0.3\% \textit{ per year}$$

$$\frac{8.6-6.4}{1990-1980} = 0.22, \textit{meaning an increase of } 0.22\% \textit{ per year}$$

$$\frac{10.2-8.6}{2000-1990} = 0.16, \textit{meaning an increase of } 0.16\% \textit{ per year}$$

$$\frac{11.1-10.2}{2008-2000} = 0.11, \textit{meaning an increase of } 0.11\% \textit{ per year}$$

Let's look at a logarithmic equation that is decreasing and find its graph:

Example 5.1.3: Graphing a Decreasing Logarithmic Equation

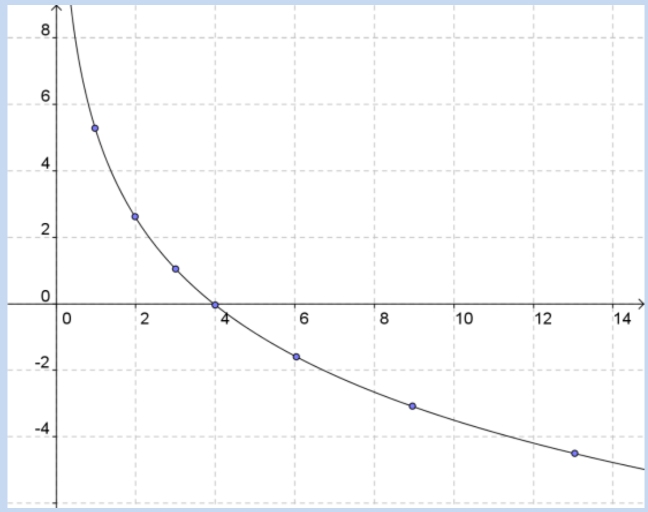
Find some ordered pairs then graph the equation: y = 5.2 - 8.7logx

Solution:

You are free to pick any positive value for x in the logarithmic equation.

A partial list of ordered pairs is shown in the table.

x	y
1	5.2
2	2.58
3	1.05
4	-.04
6	-1.57
9	-3.1
13	-4.49



y = 5.2 - 8.7logx

y = 5.2 - 8.7log(3) replace x with 3

y ≈ 5.2 - 8.7(.4771) enter log(3) into your calculator

y ≈ 5.2 - 4.1508 multiplication

y ≈ 1.05 rounded to the hundredth place

Note: Notice this logarithmic graph decreases quickly at first then levels off over time. Logarithmic equations do a good job of modeling things that level off in their decline like a dieter's progress, a river level after the rain stops, or price per unit for bulk purchases.

Mathematicians refer to the acceptable inputs "x's" in an equation to be the equation's **domain**. The domain of a logarithmic equation is all x's greater than zero.

Section 5.1: Problem Set

1. Stores commonly offer a cheaper unit price for large quantity purchases.

Quantity	Unit Price
1	100.00
2	80.00
5	70.00
10	50.00
20	40.00

   a) Make a graph of the data large enough to include a quantity of 30 (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
   b) Find the slope between quantities 1 and 2. Explain the meaning of the slope in context.
   c) Find the slope between quantities 5 and 10. Explain the meaning of the slope in context.
   d) Add a trend line to the graph.
   e) Estimate the unit price for a quantity of 30.
   f) Estimate the quantity for a unit price of $44.

2. Partycheap.com offers a bulk discount for buying more items.

Bulk Discount		
Buy...	Save...	New Price...
12 - 23	15%	$11.39ea.
24 - 71	20%	$10.72ea.
72 or more	30%	$9.38ea.

   a) Make a graph of the data (use graph paper, label completely, and use the first number in the buy range for the independent(x) variable and new price for the dependent(y) variable).
   b) Find the slope between buying 12 and 24, accurate to 3 decimal places. Explain the meaning of the slope in context.
   c) Find the slope between buying 24 and 72, accurate to 3 decimal places. Explain the meaning of the slope in context.
   d) Add a trend line to the graph.
   e) Estimate the number a customer would have to buy to pay $10ea.

3. The relationship between diameter and height of a tree is shown in the table.

Diameter(ft)	Height (ft)
1.0	41
1.5	59
1.8	69
2.0	75
3.3	101
4.0	109
4.4	112
5.0	116

a) Make a graph of the data large enough to include a 6 foot diameter tree (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
b) Find the slope between the 2 and 4 foot diameter trees. Explain the meaning of the slope in context.
c) Find the slope between the 1.5 and 4.4 foot diameter trees, round to the nearest tenth place. Explain the meaning of the slope in context.
d) Add a trend line to the graph.
e) Estimate the height of a 6 foot diameter tree.
f) Estimate the diameter of a 30 foot tall tree.

4. The Illinois River flow in cubic feet/second (CFS) is shown in the month of May as the rainy season ends and the level starts dropping.

May	Level
8	744
9	645
10	558
11	492
12	442
13	398
14	367
15	342
16	324
17	302

a) Make a graph of the data large enough to include 1000 CFS and May 22nd (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
b) Find the slope between May 8th and 12th. Explain the meaning of the slope in context.
c) Find the slope between May 14th and 16th. Explain the meaning of the slope in context.
d) Add a trend line to the graph.
e) Estimate the flow on May 22nd.
f) Estimate the date the flow was 1000 CFS.

5. Temperature and snowfall are related. The data in the table is recorded monthly for Crater Lake National Park.

Average Low Temperature (°F)	Average Snowfall (in.)
17.8	104.9
18.4	83.8
19.1	84.2
22.7	44.9
28.5	20.1
34.1	3.8
41	0.2
40.9	0.1
36.5	2.8
30.7	21.6
23.5	63.6
19.5	93.5

   a) Make a graph of the data large enough to include 15° (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
   b) Find the slope between 40.9° and 36.5°, round to the nearest tenth place. Explain the meaning of the slope in context.
   c) Find the slope between 17.8° and 18.4°, round to the nearest tenth place. Explain the meaning of the slope in context.
   d) Add a trend line to the graph.
   e) Estimate the snowfall if the temperature is 15°.
   f) Estimate the temperature if there is 72 inches of snowfall.

6. The percent of black females that are divorced is listed in the table.

Year	Percent Divorced
1970	4.4
1980	8.6
1990	10.6
2000	11.8
2008	12.7

   a) Make a graph of the data considering 1960 to be year 0 (use graph paper, label completely, and choose the correct axis for the independent(x) and dependent(y) variables).
   b) Find the slope between 1970 and 1980. Explain the meaning of the slope in context.
   c) Add a trend line to the graph.
   d) Estimate the percent you would expect to be divorced in 2014.
   e) Estimate the year you would expect to reach 14% divorced.

Section 5.2: Finding Logarithmic Equations

In this section we learn how to find a logarithmic equation from ordered pairs using regression.

Again, the equation gives you accurate answers between the data points and predictive power beyond the range of the graph.

Consider the pediatric data:

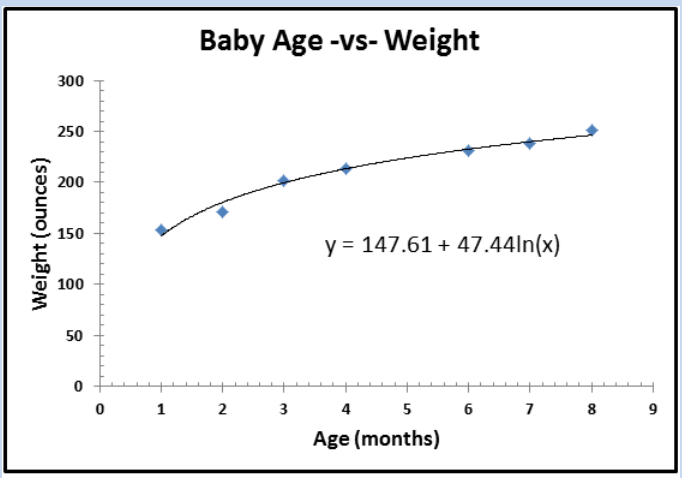
Example 5.2.1: Finding a Logarithmic Equation

The data shows the weight gain of a baby for selected months during the first year. Use the regression feature of your graphing calculator to find a logarithmic equation to model the data. Then use the equation to predict the weight at 20 months.

AGE	WEIGHT
Birth (Nov. 12, 2011)	7 pounds, 2 ounces
1 month	9 pounds, 9 ounces
2 months	10 pounds, 11 ounces
3 months	12 pounds, 10 ounces
4 months	13 pounds, 6 ounces
6 months	14 pounds, 7 ounces
7 months	14 pounds, 14 ounces
8 months	15 pounds, 11 ounces

Solution:

Enter the 7 ordered pairs into your calculator changing the weight to ounces and use logarithmic regression (LnReg) and see if you agree with the *natural logarithmic* equation listed on the graph. The weight at birth (age 0) must be left out to find a logarithmic model since the log(0) is undefined.



$y = 147.61 + 47.44\ln(x)$

$y = 147.61 + 47.44\ln(20)$ replace x with 20

$y = 147.6 + 47.44(2.9957)$ exponents first

$y = 147.61 + 142.12$ multiplication next

$y = 289.73$ addition last

Final Answer: If the pattern continues, the baby's weight will be approximately 290 ounces (or 18lbs, 2 oz.) in the 20th month.

Note: Recall from chapter 3 that power equations have the same basic shape. Regression gives a very good power model for the data as well: $y = 151.07x^{0.24}$. Replacing "x" with 20 in this equation results in a prediction of 310 ounces for the baby's weight in the 20th month.

Note: The calculator employs the *natural* logarithm (ln) instead of the *common* logarithm (log) for regression. Common logarithms have a *hidden* base of 10, natural logarithms have a *hidden* base of e ≈ 2.718.

Natural logarithms are base *e*. The symbol *e* is used in place of its messy decimal approximation (2.718281828); just as the symbol π is used in place of its messy decimal approximation (3.14159265).

Without sticking our head in so deep into the rabbit hole that we fall in … here is a brief explanation of the origin of *e*:

Consider the exponential model for \$100 in savings growing *annually* at 6% for 5 years.

The amount (A) it would grow to would be:

$A = 100(1 + .06)^5 \approx \$133.82$ … recall the equation $y = ab^x$ from chapter 4 where "a" is the initial value and "b" is the growth rate.

If your \$100 grew *monthly*, 6% would have to be divided by 12 to find the monthly interest rate (.06/12 = .005), and instead of growing for 5 years, it would grow for 60 months:

$A = 100(1 + .005)^{60} \approx \$134.89$ … you make more money if it is compounded more often.

If your dollar grew *daily*, 6% would have to be divided by 365 to find the daily interest rate (.06/365 ≈ .00016438), and instead of growing for 5 years, it would grow for 1825 days:

$A = 100(1 + .00016438)^{1825} \approx \$134.98$

In nature plants, animals, and populations grow ***continuously*** not yearly, monthly or even daily. Watch what happens when the growth we add to 100% gets smaller while the number of times it grows gets larger:

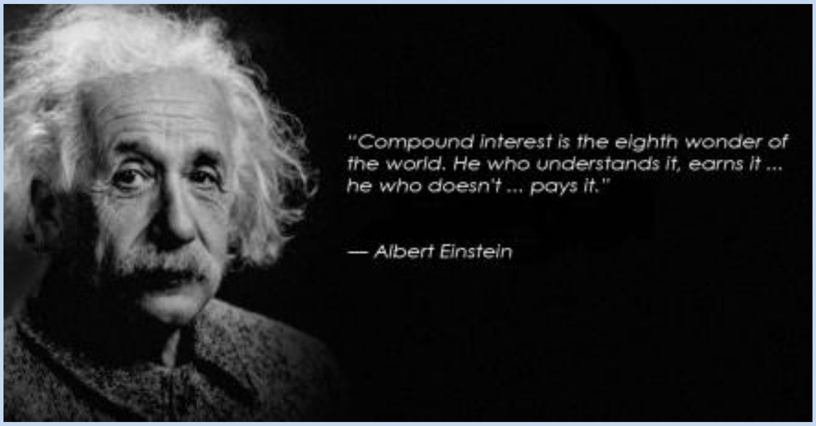
$$\left(1+\tfrac{1}{8}\right)^{8} \approx 2.5658$$

$$\left(1+\frac{1}{20}\right)^{20} \approx 2.6533$$

$$\left(1+\frac{1}{50}\right)^{50} \approx 2.6916$$

$$\left(1+\frac{1}{3000}\right)^{3000} \approx 2.7178$$

$$\left(1+\frac{1}{400000}\right)^{400000} \approx 2.7183$$



If we let the number we are changing go to infinity we get this curious decimal *e* that, like *π*, goes on forever without any discernable pattern.

This discovery led to an exponential equation with *e* as its base which is used to model continuous growth. Most things grow continuously, so your calculator uses the natural logarithm for regression.

Continuous Exponential Growth

$A = A_0e^{rt}$ models continuous growth; where A_0 = initial amount, r = rate, and t = time.

Reconsider the previous compound interest example now using the continuous exponential growth model:
$A = 100e^{(.06 \cdot 5)} \approx \$134.99$ … this is the limit we saw it approaching as we compounded the interest more often.

This equation models anything that is growing (or declining) *continuously*: weeds, possums, city populations, home values, river levels, bacteria, and even the spread of disease.

Since *e* shows up regularly in *nature* in the base of exponential equations, it made sense to create a logarithm with *e* as its base to save time … appropriately named the natural logarithm.

Natural Logarithm

$\ln x = \log_e x$

Consider the coffee prices:

Example 5.2.2: Finding a Logarithmic Equation

The prices are for the Human Bean Granita, really just an expensive and delicious chocolate Slurpee.
Use the regression feature of your graphing calculator to find a logarithmic equation to model the price/ounce as a function of the size. Then use the equation to come up with a price for their 32 ounce size.

Size (oz.)	Price
8	\$2.25
12	\$2.75
16	\$3.25
20	\$3.75

Solution:
Divide price by ounces (i.e. \$3.25/16oz. ≈ \$.2031/oz. or 20.31¢/oz.).
Entering the 4 ordered pairs into your calculator and using logarithmic regression (LnReg) returns the equation:
$y = 49.15 - 10.31\ln x$; where x = the ounces and y = the price per ounce.

Size (oz.)	Price/oz.
8	28.13¢/oz.
12	22.92¢/oz.
16	20.31¢/oz.
20	18.75¢/oz.

$y = 49.15 - 10.31\ln x$

$y = 49.15 - 10.31\ln(32)$ replace x with 32

$y \approx 49.15 - 10.31 \bullet 3.466$ exponents first (logarithms are a form of an exponent)

$y \approx 49.15 - 35.73$ multiplication next

$y \approx 13.42$ subtraction last

13.42¢/oz. • 32 oz. = 429¢

Final Answer: Modeling the decrease in price per ounce with a natural logarithmic equation would suggest a price of \$4.25 or \$4.50 since they like to round their prices to the quarter.

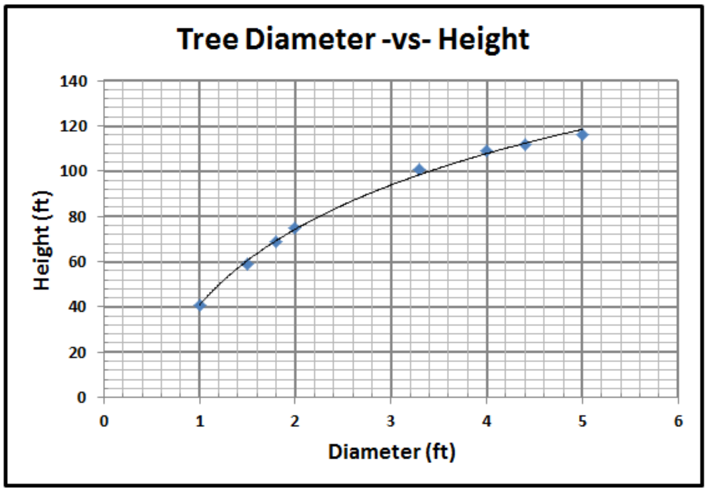
Note: Interestingly, their price is presently \$6.00 for this drink which is 18.75¢/ounce, so there is no incentive to choose the larger size over the 20 oz. size.

Section 5.2: Problem Set

1. The relationship between diameter and height of a particular tree is shown in the table and graph.

Diameter(ft)	Height (ft)
1.0	41
1.5	59
1.8	69
2.0	75
3.3	101
4.0	109
4.4	112
5.0	116

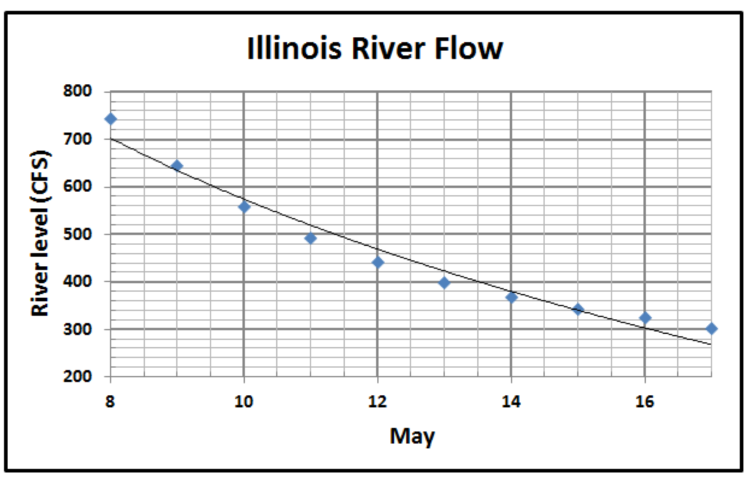
   a) Use regression to find a logarithmic equation to model the data. Round the numbers in your equation to 2 decimal places.
   b) Use your equation to estimate the height of a tree with a diameter of 6 feet, accurate to 1 decimal place.
   c) Use your equation to estimate the height of a tree with a diameter of 8 feet, accurate to 1 decimal place.



2. The Illinois River flow in cubic feet/second (CFS) is shown in the month of May as the rainy season ends and the level starts dropping.

May	Level
8	744
9	645
10	558
11	492
12	442
13	398
14	367
15	342
16	324
17	302

   a) Use regression to find a logarithmic equation to model the data. Round the numbers in your equation to 2 decimal places.
   b) Use your equation to estimate the level on May 2, accurate to the nearest cubic foot.
   c) Use your equation to estimate the level on May 25, accurate to the nearest cubic foot.



3. Consider the price list for rectangular tarps of different sizes.

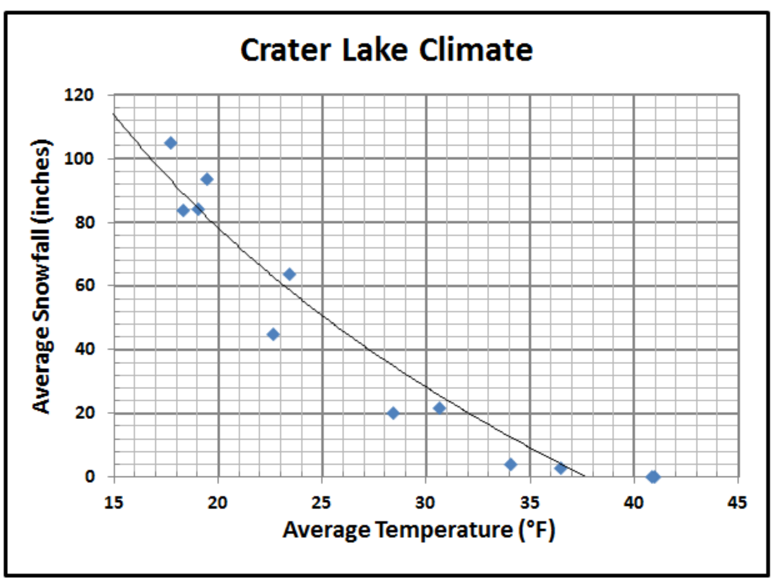
   a) Use regression to find a logarithmic equation to model the data using area as the "x" variable and price per square foot as the "y" variable. Round the numbers in your equation to 3 decimal places.
   b) Use your equation to determine the right price for the 2 larger tarps in the table. Round to the nearest dollar.

Dimensions	Area	Price	Price Per Square Foot
5' x 7'		$4	
6' x 8'		$5	
8' x 10'		$8	
10' x 12'		$11	
12' x 14'		$15	
16' x 20'		$26	
20' x 24'			
24' x 30'			

4. Temperature and snowfall seem like they would be related. The data in the table is recorded monthly for Crater Lake National Park.

   a) Use regression to find a logarithmic equation to model the data. Round the numbers in your equation to 2 decimal places.
   b) Use your equation to predict the snowfall for a month with an average temperature of 25°, rounded to the nearest inch.
   c) Use your equation to predict the snowfall for a month with an average temperature of 15°, rounded to the nearest inch.

Average Low Temperature (°F)	Average Snowfall (in.)
17.8	104.9
18.4	83.8
19.1	84.2
22.7	44.9
28.5	20.1
34.1	3.8
41	0.2
40.9	0.1
36.5	2.8
30.7	21.6
23.5	63.6
19.5	93.5



5. Stores commonly offer a cheaper unit price for large quantity purchases.

Quantity	Unit Price
1	100.00
2	80.00
5	70.00
10	50.00
20	40.00

   a) Use regression to find a logarithmic equation to model the data. Round the numbers in your equation to 2 decimal places.
   b) Use your equation to find an appropriate unit price for a customer who purchases 15 items.
   c) Use your equation to find an appropriate unit price for a customer who purchases 25 items.

6. Partycheap.com offers a bulk discount for buying in larger quantities. The owner decided to make a new price list giving a different discount for any number purchased.

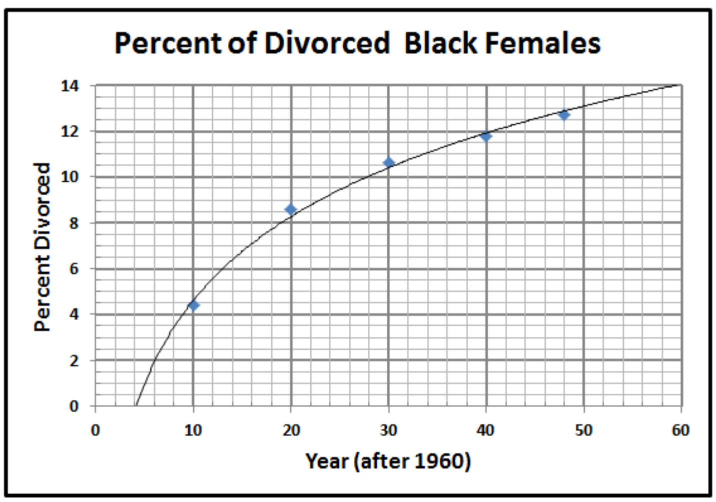
   a) Use regression to find a logarithmic equation to model the data. Use the first number in the buy range for the independent(x) variable and new price for the dependent(y) variable Round the numbers in your equation to 2 decimal places.
   b) Use your equation to find the new price for a customer who purchases 20 items.
   c) Use your equation to find the new price for a customer who purchases 40 items.
   d) Use your equation to find the new price for a customer who purchases 1 item.
   e) The chart lists $11.39 as the price following a 15% discount. Calculate the original price per item from which the discounts are taken.

Bulk Discount		
Buy...	**Save...**	**New Price...**
12 - 23	15%	$11.39ea.
24 - 71	20%	$10.72ea.
72 or more	30%	$9.38ea.

7. The percent of divorced black women over the age of 15 is recorded in the table from 1970 through 2008. Consider the data as years past 1960.

Year	Percent Divorced
10	4.4
20	8.7
30	11.2
40	11.8
48	12.7

   a) Use regression to find a logarithmic equation to model the data. Round the numbers in your equation to 2 decimal places.
   b) Use your equation to predict the divorce rate in 2015, accurate to the nearest tenth of a percent.
   c) Use your equation to predict the divorce rate in 2020, accurate to the nearest tenth of a percent.



Section 5.3: Using Logarithmic Equations

Consider the growth curve for the female California Halibut:

Example 5.3.1: Estimating Age

The data models the age versus length of the female halibut in California.

Age (yrs)	Length (mm)
3.6	390
5.7	575
12.6	920

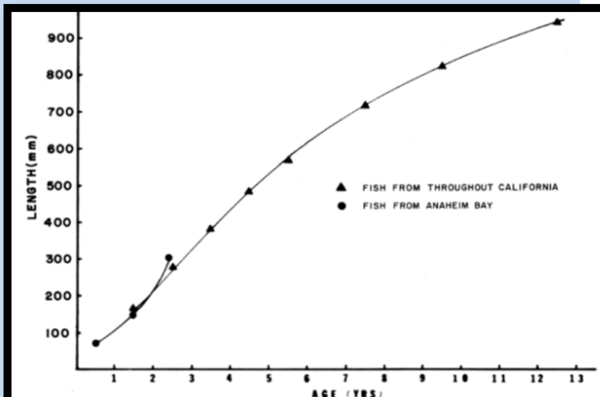


FIGURE 43—Comparison of growth curves of female California halibut from Anaheim Bay, and from throughout California (Frey, 1971).

Use the regression feature of your graphing calculator to find a logarithmic model for the data. Then use the equation to estimate the age of an 1100 mm female halibut caught by a biologist.

Solution:

Enter the 3 ordered pairs into your calculator to obtain the equation: $y = -157.47 + 424.37\ln x$

$y = -157.47 + 424.37\ln x$

$1100 = -157.47 + 424.37\ln x$ replace y with 1100

$1257.47 = 424.37\ln x$ add 157.47 to both sides

$2.963 \approx \ln x$ divide both sides by 424.37

$e^{2.963} \approx x$ switch to exponential form

$x \approx 19.4$ enter e^2.963 in your calculator (e^x is located above the LN button)

Final Answer: The female halibut is approximately 19 years old.

Note: Recall from chapter 3 that power equations have the same basic shape. Regression gives a power model for the data as well: $y = 169.34x^{0.675}$; Replacing y with 1100 in this equation results in a prediction of about 16 years of age.

Recall from section 4.3 that the logarithmic expression $x = \log_4 12$ was introduced to solve the exponential equation $4^x = 12$. We used the same principle in the reverse direction in the previous example. $2.963 = \log_e x$, so this also must be solving the equation $e^{2.963} = x$. Formally then, you can convert back and forth from logarithmic to exponential equations as needed:

Converting back and forth from Logarithmic to Exponential

Logarithmic form: $a = \log_b c$

Exponential form: $b^a = c$

Consider the data recorded on glacial width:

Example 5.3.2: Glacial Retreat

The data shows the width of a glacier measured in meters over a 30 year period.

Use the regression feature of your graphing calculator to find a logarithmic model for the data, considering 1975 to be year 5. Then use the equation to predict the year the glacier will be down to 150 meters wide.

Year	Width (m)
1975	235
1980	210
1985	196
1990	186
1995	178
2000	171
2005	166

Solution:

Enter the 7 ordered pairs into your calculator to obtain the equation: $y = 291.96 - 35.45\ln x$

$y = 291.96 - 35.45\ln x$

$150 = 291.96 - 35.45\ln x$ replace y with 150

$-141.96 = -35.45\ln x$ subtract 291.96 from both sides

$4 \approx \ln x$ divide both sides by -35.45

$e^4 \approx x$ switch to exponential form

$x \approx 55$ enter e^4 in your calculator

Final Answer: The year would be 1970 + 55 or 2025.

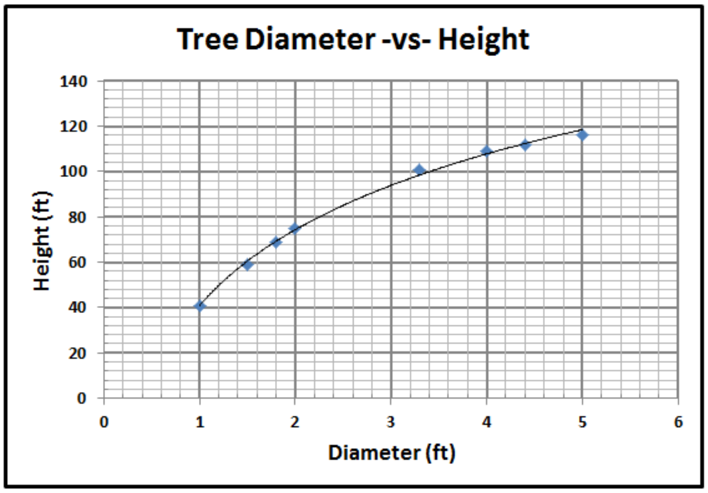
The Easton Glacier retreated 255 m (837 ft) from 1990 to 2005.

Section 5.3: Problem Set

1. The relationship between diameter and height of a particular tree in is shown in the table and graph.

Diameter(ft)	Height (ft)
1.0	41
1.5	59
1.8	69
2.0	75
3.3	101
4.0	109
4.4	112
5.0	116

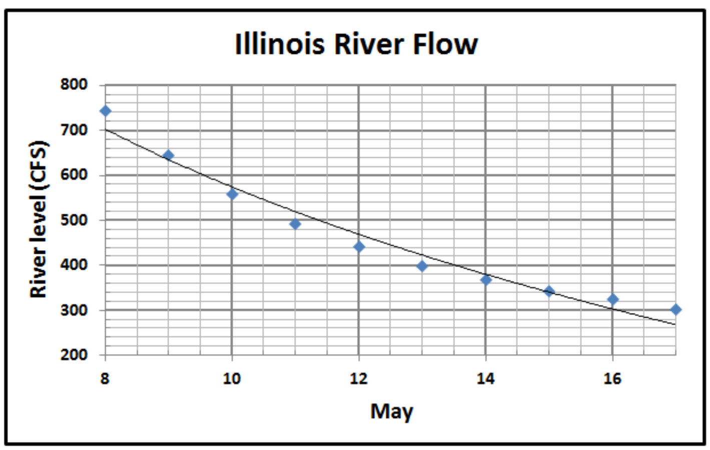
   a) Use regression to find a logarithmic equation to model the data. Round the numbers in your equation to 2 decimal places.
   b) Use your equation to calculate the diameter a tree should have to be 90 feet tall, accurate to the tenth place.
   c) Use your equation to calculate the diameter a tree should have to be 150 feet tall, accurate to 1 decimal place.



2. The Illinois River flow in cubic feet/second (CFS) is shown in the month of May as the rainy season ends and the level starts dropping.

May	Level
8	744
9	645
10	558
11	492
12	442
13	398
14	367
15	342
16	324
17	302

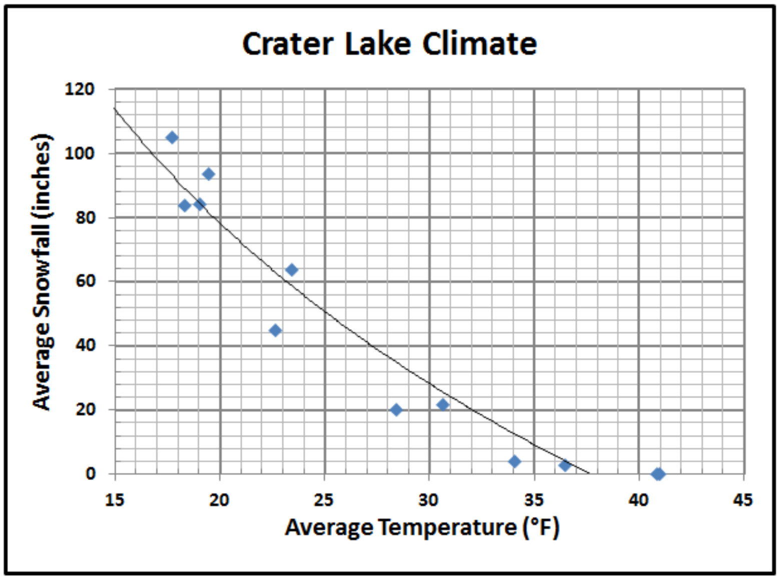
   a) Use regression to find a logarithmic equation to model the data. Round the numbers in your equation to 2 decimal places.
   b) Use your equation to calculate the date the level will drop to 200 CFS, accurate to the nearest tenth place.
   c) Use your equation to calculate the date the level was 1000 CFS, accurate to 1 decimal place.



3. The data in the table is recorded monthly for Crater Lake National Park.

Average Low Temperature (°F)	Average Snowfall (in.)
17.8	104.9
18.4	83.8
19.1	84.2
22.7	44.9
28.5	20.1
34.1	3.8
41	0.2
40.9	0.1
36.5	2.8
30.7	21.6
23.5	63.6
19.5	93.5

   a) Use regression to find a logarithmic equation to model the data. Round the numbers in your equation to 2 decimal places.
   b) Use your equation to predict the temperature necessary to produce 50 inches of snowfall, accurate to 1 decimal place.
   c) Use your equation to predict the temperature necessary to produce 120 inches of snowfall, accurate to 1 decimal place.



4. Stores commonly offer a cheaper unit price for large quantity purchases.

Quantity	Unit Price
1	100.00
2	80.00
5	70.00
10	50.00
20	40.00

   a) Use regression to find a logarithmic equation to model the data. Round the numbers in your equation to 2 decimal places.
   b) Use your equation to find an appropriate quantity for a customer who wishes to get a $35 unit price, accurate to the nearest whole number.
   c) Use your equation to find an appropriate quantity for a customer who wishes to get a $60 unit price, accurate to the nearest whole number.

5. Partycheap.com offers a bulk discount for buying in larger quantities. The owner decided to make a new price list giving a different discount for any number purchased.

   a) Use regression to find a logarithmic equation to model the data. Use the first number in the buy range for the independent(x) variable and new price for the dependent(y) variable Round the numbers in your equation to 3 decimal places.

   b) Use your equation to find the amount a customer should be asked to buy to get a price of $10 each, accurate to the nearest whole number.

   c) Use your equation to find the amount a customer should be asked to buy to get a price of $9 each, accurate to the nearest whole number.

Bulk Discount		
Buy...	**Save...**	**New Price...**
12 - 23	15%	$11.39ea.
24 - 71	20%	$10.72ea.
72 or more	30%	$9.38ea.

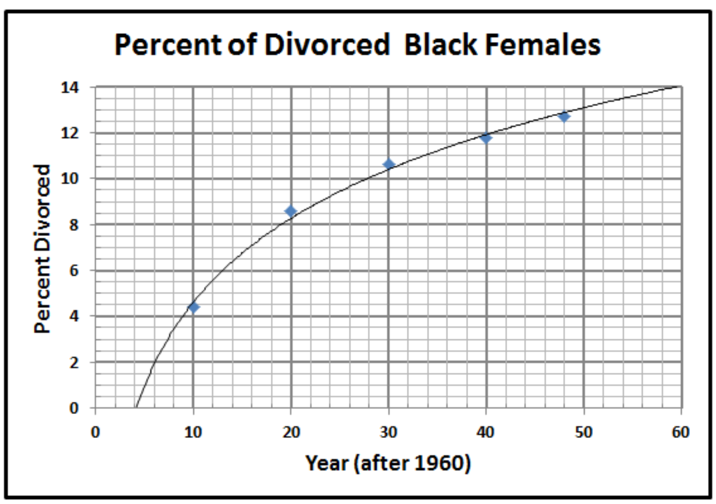
6. The percent of divorced black women over the age of 15 is recorded in the table from 1970 through 2008. Consider the data as years past 1960.

Year	Percent Divorced
10	4.4
20	8.7
30	11.2
40	11.8
48	12.7

   a) Use regression to find a logarithmic equation to model the data. Round the numbers in your equation to 3 decimal places.

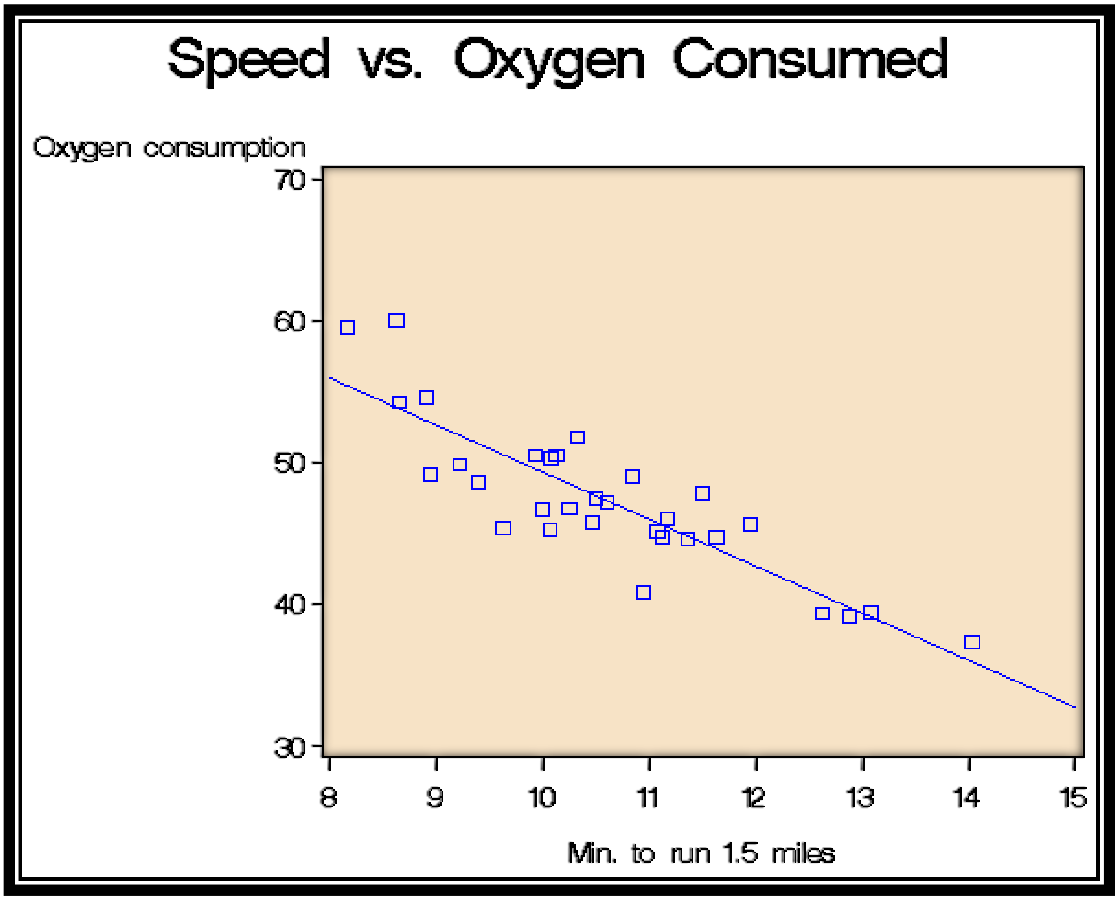
   b) Use your equation to predict the year the divorce rate will reach 14%, accurate to 1 decimal place.

   c) Use your equation to estimate the year the divorce rate would have been 2%, accurate to the nearest tenth place.



Chapter 6:

Choosing the Right Model



http://v8doc.sas.com/sashtml/analyst/chap1/sect6.htm

Now the training wheels come off and we are left to find our own balance as to which of the five equations we have studied will be the best model.

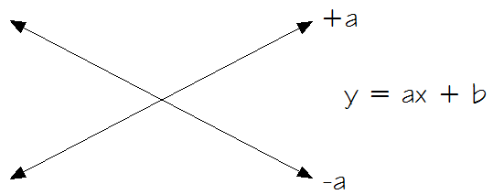
We will need a good mix of training and practice.

Section 6.1: Compliant Data

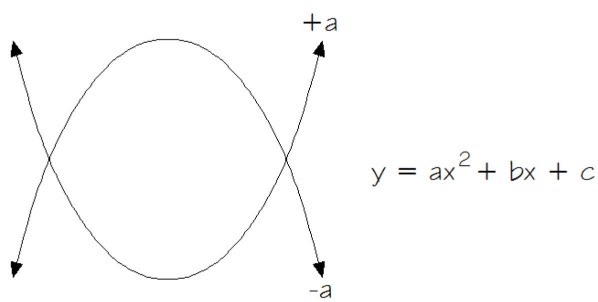
We have studied five of the basic functions in the preceding chapters and looked at data that reasonably fit each type of function. This section will give you some practice at choosing the correct model for data when the choice is up to you.

Recall the shapes that we have observed and their equations, and note the effect of the negative coefficient on the shape:

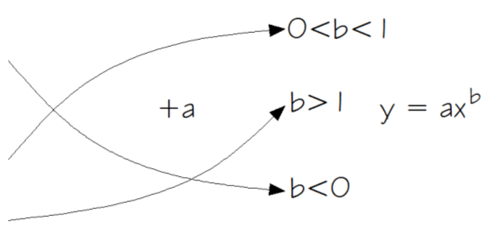
Linear:



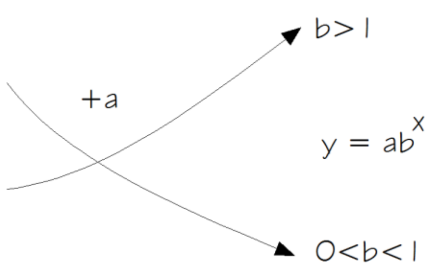
Quadratic:



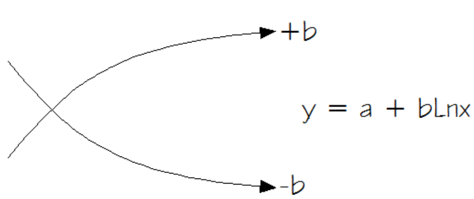
Power:



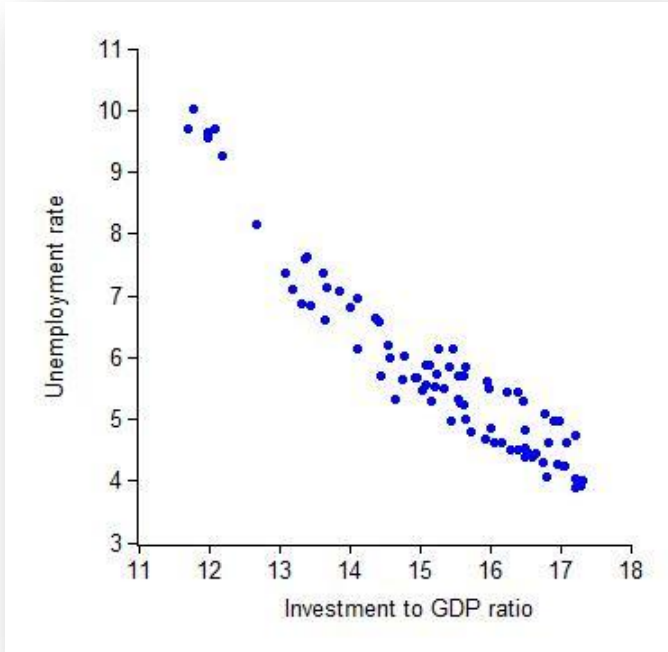
Exponential:



Logarithmic:



Notice that the relationship between investment and unemployment shown in the graph could be reasonably modeled by any of the functions we have studied, with the exception of an exponential function.



http://gregmankiw.blogspot.com/2011/03/striking-scatterplot.html

Ultimately choosing the correct equation to model data is a dynamic process … it is not an exact science! There are some basic guidelines, however, that should be considered that can be easily memorized with the mnemonic ERC.

Choosing the right equation to model data is based on 3 considerations (ERC):

1. **E**yeball test: Observing the "fit" of the graph with the ordered pairs in STATPLOT on the graphing calculator.
2. **R**-value: Which equation has the largest correlation coefficient?
3. **C**ommon sense: You may know something about the relationship that would affect your choice.

CHOOSING THE BEST MODEL

1. **Eyeball Test**
2. **R-value**
3. **Common Sense**

Consider the data regarding the effect of government spending:

Example 6.1.1: Government Investment -vs- Unemployment

The data shows the effect of government investment on the unemployment rate.

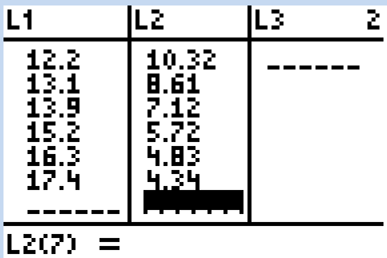
Use the features of your graphing calculator to find the best model for the data. Then use the equation to predict the investment ratio required to get the unemployment rate to 3%.

Investment to GDP Ratio	Unemployment Rate (%)
12.2	10.32
13.1	8.61
13.9	7.12
15.2	5.72
16.3	4.83
17.4	4.34

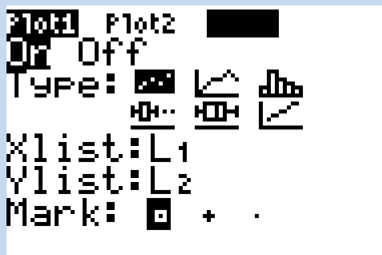
Solution:

1. **The eyeball test:**

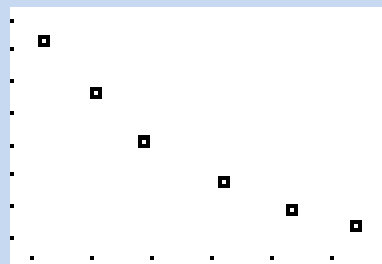
Enter the data in the STAT button



Set up STAT PLOT as shown



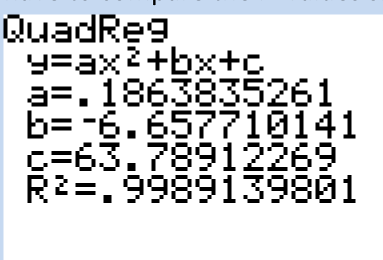
ZoomStat behind the ZOOM button

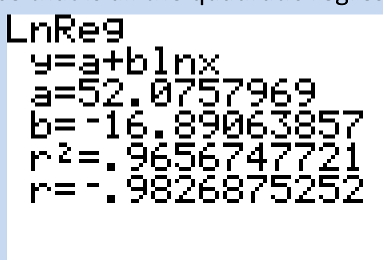


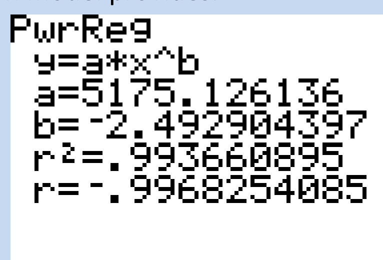
The eyeball test rules out the linear and exponential functions.

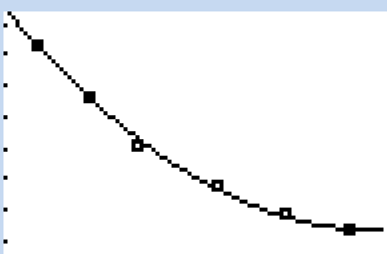
2. **R-value:**

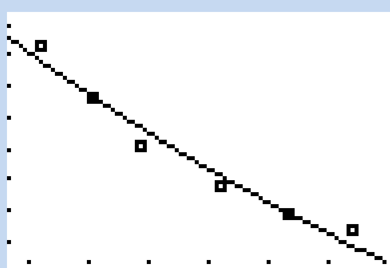
The r-value (and eyeball) would favor the quadratic model slightly over the power model. Notice you have to compare the r^2 values since that is all the quadratic regression model provides.

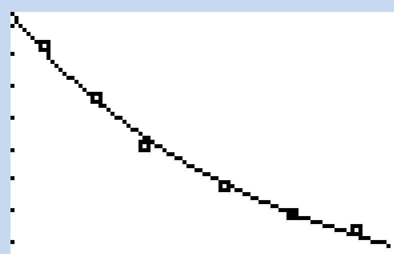












3. **Common sense:**

The quadratic model does not pass the common sense test since it begins increasing when viewed in a larger window.

Final Answer: The power model has the best r-value.
Substituting 3 for "y" in the power equation: $y = 5175.13x^{-2.49}$ and solving for "x", gives an investment ratio of 19.9.

Consider the health data:

Example 6.1.2: Spread of a Disease

The data from the center of disease control (CDC) shows the number of cases of the flu in a region where a vaccination program was implemented.

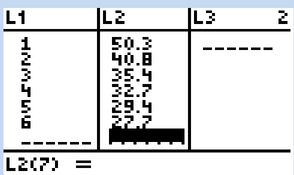
Use the features of your graphing calculator to find the best model for the data. Then use the equation to predict the number of cases in 2016 and to estimate the year there will be 20,000 cases.

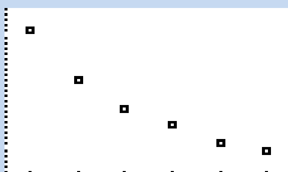
Year	Flu Cases (thousands)
2008	50.3
2009	40.8
2010	35.4
2011	32.7
2012	29.4
2013	27.7

Solution:

Where the x-values are dates (or any large number) it is usually wise to choose something smaller.
Choosing 2008 as year 0 is a bad idea because logarithmic and power equations will not work.
Choosing 2000 as year 0 makes 2008 conveniently an 8 … this is a good idea.
Choosing 2008 as year 1 will be a better fit if the data has a steeper slope in the beginning … as it is here.

1. **The eyeball test:**

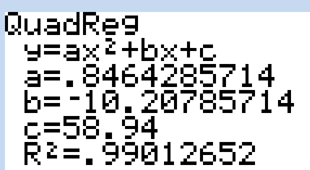


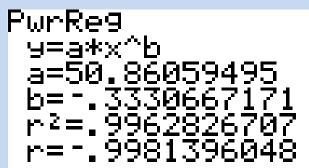


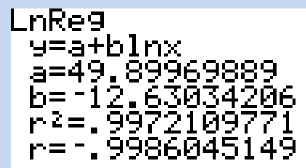
Entering the data with 2008 as year 1, the eyeball test would suggest quadratic, power or logarithmic as possible options.

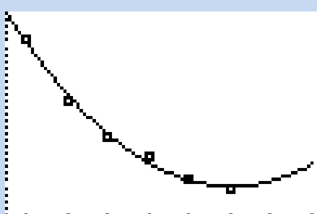
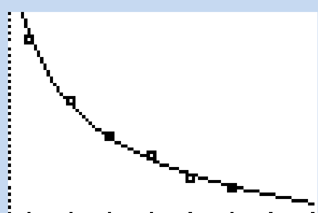
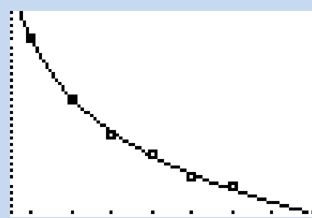
2. **R-value:**

The r-value would favor the logarithmic model slightly over the power model.



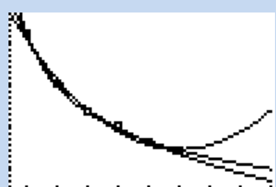
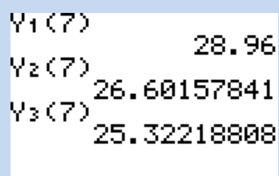




3. **Common sense:**

It is truly a toss-up here. The question of what will happen in the future is anyone's guess. The data for 2014 (year 7) will figure prominently as to which model used in 2013 was actually more accurate. The models diverge by thousands from one another in their predictions even for one year.

Final Answer: In the absence of any other considerations, the logarithmic model has the best r-value.
Substituting 9 for x into y = 49.8997 - 12.6303ln(x) and solving for "y", predicts 22,148 cases in 2016.
Substituting 20 for y into y = 49.8997 - 12.6303ln(x) and solving for "x", predicts the year 2017.7.

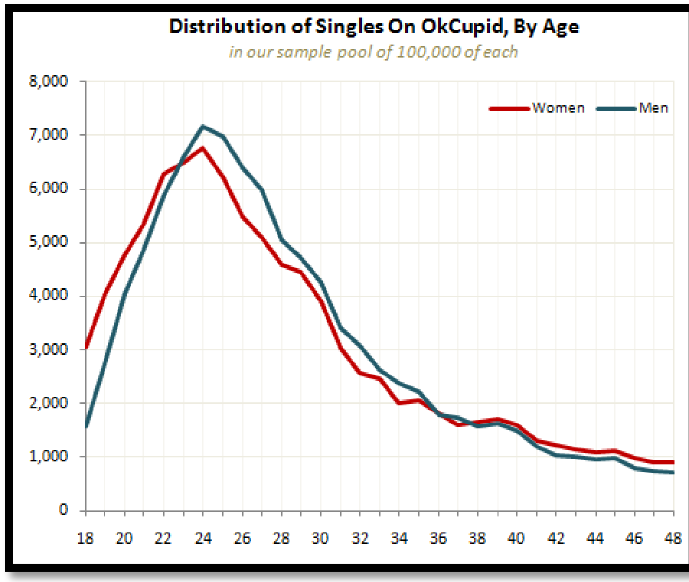
Note: One could use the first 5 points to find an equation then see which does the best job of predicting 2013, since we know it. This approach would have slightly favored the power model, which predicted it 200 cases more accurately than the logarithmic model.

Section 6.1: Problem Set

1. The number of single men in a sample taken from the OkCupid dating site is listed in the table as a function of age.

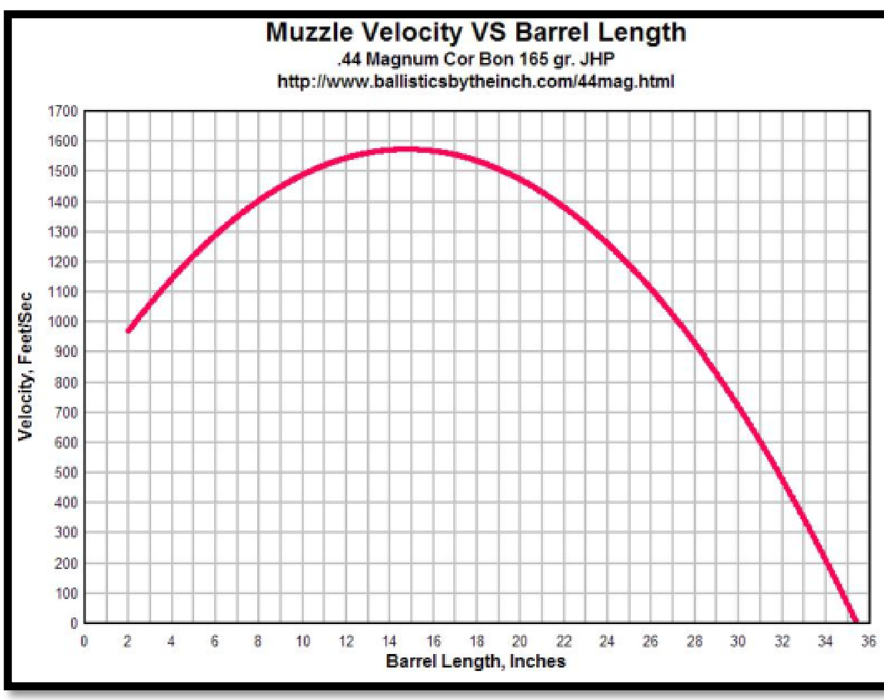
Age	Number
27	6000
34	2400
46	1000

   a) Use regression and the graph of the data to find the best model for the decreasing side of the graph. (The curve for the men peaks over 7,000)
   b) Briefly defend the equation you chose to model the data.
   c) Use your equation to estimate the number of single men you would expect at ages 30 and 40 and compare to the number represented on the graph.



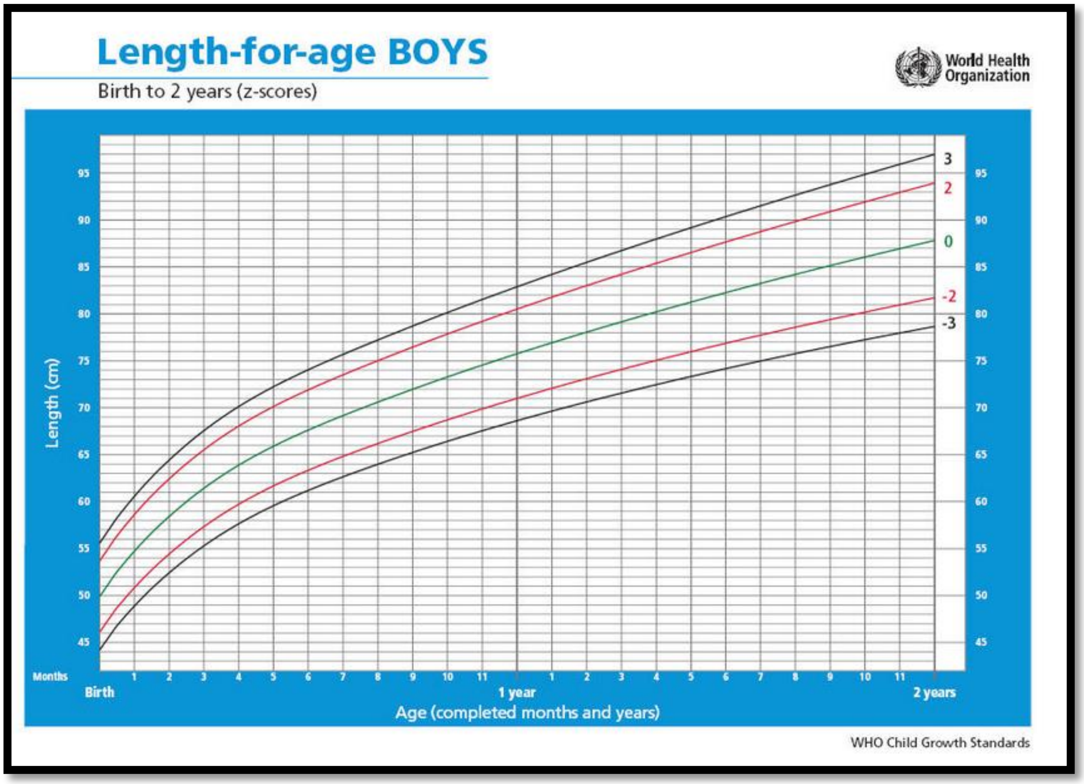
2. The graph shows how barrel length effects muzzle velocity for a .44 magnum bullet.

   a) Choose some points off the graph and use regression and the graph of the data to find the best model for the data.
   b) Briefly defend the equation you chose to model the data.
   c) Compare the vertex that your equation would specify with the graph (see section 2.3).



3. The growth curve for boys used by health care professionals is shown below. The center curve represents the average length and the outer curves pertain to 2 and 3 standard deviations from this average (a statistics course will teach you what to make of standard deviation).

   a) Identify some ordered pairs from the center graph and use regression and the graph of the data to find the best model for the data.
   b) Briefly defend the equation you chose to model the data.
   c) Use your equation to calculate to predict the length a boy "should" be at 36 months.



4. The flow during the month of May is shown for the Chetco River in Oregon in 2014.

   a) Use regression and the graph of the data to find the best model for the data.
   b) Briefly defend the equation you chose to model the data.
   c) Use your equation to predict the flow on June 9th.
   d) Use your equation to predict the day the flow will hit 345 CFS.
   e) Find the slope between May 10th and 14th and explain its meaning in context.

May	Flow (CFS)
10	3430
12	2230
14	1670
16	1370
18	1260
20	1080
22	963
24	868
26	787
28	720
30	657

5. The number of payday loan calls received by National Debtline is shown in the table over a recent 5 year span.

Year	Number
2007	480
2008	860
2009	1300
2010	3,500
2011	10,000
2012	20,000

a) Use regression and the graph of the data to find the best model for the data, considering 2007 to be year 1.
b) Briefly defend the equation you chose to model the data.
c) Use your equation to estimate the number payday loans that could be expected in 2014.
d) Use your equation to predict the year the number of payday loan calls will reach 50,000.

6. Prices for drip coffee at a drive through stand are shown in the table.

Size (oz.)	Price
8	$2.00
12	$2.50
16	$3.00
20	$3.50
24	$3.75

a) Use regression and the graph of the data to find the best model for the data.
b) Briefly defend the equation you chose to model the data.
c) Use your equation to set a price for a 32 ounce coffee, rounded to the nearest quarter.

7. Prices for the Java Chip Smoothie at the Human Bean are shown in the table.

Size (oz.)	Price	Price(¢)/oz.
8	$3.00	
12	$3.50	
16	$4.00	
20	$4.50	

a) Calculate the price per ounce for each size .
b) Use regression and the graph of the data to find the best model for the data (size versus price/oz.).
c) Briefly defend the equation you chose to model the data.
d) Use your equation to set a fair price for a 32 ounce smoothie, rounded to the nearest 25¢.

8. The table shows how the field strength (measured in Gauss) drops off with distance for a Samarium Cobalt Grade 18 disc magnet (1 inch in diameter and ½ inch long).

Distance (inches)	Strength (Gauss)
.0625	2690
.125	2320
.250	1660
.375	1160
.500	810
.625	580
.750	420
.875	310
1.000	240

a) Use regression and the graph of the data to find the best model for the data.
b) Briefly defend the equation you chose to model the data.
c) Use your equation to predict the field strength at 1.15 inches.
d) Use your equation to find the distance for a field strength of 1000 Gauss.
e) Give and explain the meaning of the x-intercept.

9. The maximum discharge of some of the rivers in Kansas measured in cubic feet per second (CFS) is shown relative to their contributing drainage areas.

Drainage Area (square miles)	Maximum Flow (CFS 1000's)
.66	1.34
.92	1.87
1.45	2
1.65	2.44
2.06	7.08
4.09	10.2
8.85	20.5
41	40
92	90
129	107
181	128

a) Use regression and the graph of the data to find the best model for the data.
b) Briefly defend the equation you chose to model the data.
c) Use your equation to predict the maximum flow for a river that drains 100 square miles.
d) Use your equation to predict the drainage area for a river with a maximum flow of 150,000 CFS.

10. The table shows the gallons per minute (GPM) that various tank-less water heaters can produce based on the degree that the temperature must be raised.

    a) Use regression and the graph of the data to find the best model for the NR 66 Series for the GPM as a function of the temperature.
    b) Briefly defend the equation you chose to model the data.
    c) Use your equation to predict the temperature rise you will get if the tank produces 10 GPM.
    d) Find the slope between 40^o and 50^o and explain its meaning in context.

Temp Rise (˚F)	Gallons per Minute (GPM)					
	NR111 (NC250) Series	NRC111 (NCC199) Series	NR98 (NC199) Series	NR83 Series	NR71 Series	NR66 Series
30	11.1	11.1	9.8	8.3	7.1	6.6
35	11.1	10.6	9.2	8.3	7.1	6.6
40	10.5	9.3	8.4	7.6	7.1	5.7
45	9.3	8.4	7.5	6.7	6.3	5.3
50	8.4	7.4	6.7	6.1	5.8	4.6
55	7.6	6.8	6.1	5.5	5.2	4.2
60	7.0	6.2	5.6	5.0	4.8	3.8
65	6.5	5.8	5.2	4.7	4.4	3.5
70	6.0	5.3	4.8	4.3	4.1	3.3
75	5.6	5.0	4.5	4.0	3.8	3.1
80	5.3	4.6	4.2	3.8	3.6	2.9
85	4.9	4.4	4.0	3.6	3.4	2.7
90	4.7	4.1	3.7	3.4	3.2	2.6
95	4.4	3.9	3.5	3.2	3.0	2.4
100	4.2	3.7	3.4	3.0	2.9	2.3

11. The table shows the natural gas flow in thousands of BTU's/hour for different pipe lengths and diameters. The longer the pipe the more inhibited the flow of natural gas because of the increase in friction.

    a) Consider the 1 inch pipe and use regression and the graph of the data to find the best model where the flow is a function of the pipe length.
    b) Briefly defend the equation you chose to model the data.
    c) Use your equation to predict the flow rate for 250 feet of pipe.
    d) Use your equation to find the maximum length of pipe allowable if you need 210,000 BTU's/hour of gas flow in a 1 inch pipe.

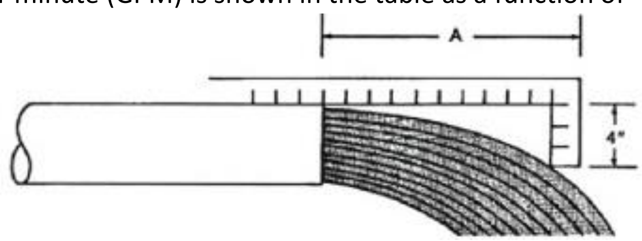
Natural Gas Piping Size Chart (From Aaladin)									
Length of Pipe in Feet	Size of Pipe in Inches								
	1/2"	3/4"	1"	1-1/4"	1-1/2"	2"	2-1/2"	3"	4"
10	108	230	387	793	1237	2259	3640	6434	
20	75	160	280	569	877	1610	2613	5236	9521
30	61	129	224	471	719	1335	2165	4107	7859
40	52	110	196	401	635	1143	1867	3258	6795
50	46	98	177	364	560	1041	1680	2936	6142
60	42	89	159	336	513	957	1559	2684	5647
70	38	82	149	317	476	896	1447	2492	5250
80	36	76	140	239	443	840	1353	2315	4900
90	33	71	133	275	420	793	1288	2203	4667
100	32	68	126	266	411	775	1246	2128	4518
125	28	60	117	243	369	700	1143	1904	4065
150	25	54	105	215	327	625	1008	1689	3645
175	23	50	93	196	303	583	993	1554	3370
200	22	47	84	182	280	541	877	1437	3160
300	17	37	70	145	224	439	686	1139	2539

12. The table is used to size home water systems. It shows the feet of head (height of the source above the faucet) that are necessary for different pounds per square inch (PSI) of pressure at the faucet.

POUNDS PRESSURE	FEET OF HEAD
1	2.31
2	4.62
3	6.93
4	9.24
5	11.60
6	13.90
7	16.20
8	18.50
9	20.80
10	23.10
11	25.40
12	27.70

a) Use regression and the graph of the data to find the best model where the feet of head is a function of pressure.
b) Briefly defend the equation you chose to model the data.
c) Use your equation to predict the pressure if there are 60 feet of head.
d) Find the slope between 7 and 11 feet and explain its meaning in context.

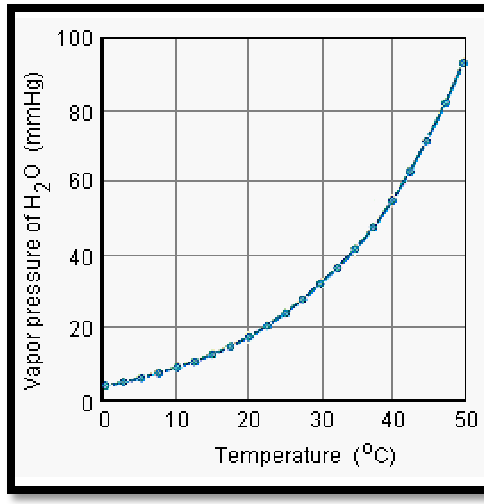
13. The discharge rate of water in gallons per minute (GPM) is shown in the table as a function of the horizontal discharge distance (A) in inches for various pipe diameters.



a) Use regression and the graph of the data to find the best model for the discharge rate as a function of A for the 1¼ inch diameter pipe.
b) Briefly defend the equation you chose to model the data.
c) Use your equation to predict the discharge rate for a horizontal discharge of 20 inches.

Horizontal Distance (A) Inches	1"	1 1/4"	1 1/2"	2"
4	8.70	9.80	13.30	22.00
5	7.10	12.20	16.60	27.50
6	8.50	14.70	20.00	33.00
7	10.00	17.10	23.20	38.50
8	11.30	19.60	26.50	44.00
9	12.80	22.00	29.80	49.50
10	14.20	24.50	33.20	55.50
11	15.60	27.00	36.50	60.50
12	17.00	29.00	40.00	66.00
13	18.20	31.50	43.00	71.50
14	20.00	34.00	46.50	77.00
15	21.30	36.30	50.00	82.50
16	22.70	39.00	53.00	88.00
17		41.50	56.50	93.00
18			60.00	99.00

14. The chart shows the relationship between temperature and vapor pressure.



a) Choose some points and use regression and the graph of the data to find the best model for the data.

b) Briefly defend the equation you chose to model the data.

c) Use your equation to generate 2 more ordered pairs and compare them to the graph.

15. The table shows the discharge in gallons per minute (GPM) for different pipe diameters as a function of feet of head. Head is the height difference between the water source and its outflow.

a) Consider feet of head as (x) and GPM as (y) and use regression and the graph of the data to find the best model for the 1¼” diameter pipe (common for residential applications).

b) Briefly defend the equation you chose to model the data.

c) Use your equation to estimate the discharge for a 1¼” diameter pipe with 144 feet of head.

d) Use your equation to estimate the feet of head necessary for a 1¼” diameter pipe to discharge 200 GPM.

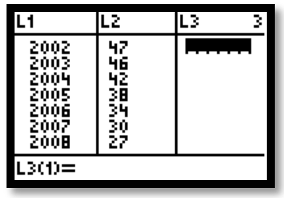
HEAD		Velocity of Discharge	DIAMETER OF NOZZLE IN INCHES								
Pounds	Feet	Feet Per Second	1	1 1/8	1 1/4	1 3/8	1 1/2	1 3/4	2	2 1/4	2 1/2
10	23.10	38.60	94.50	120.00	148.00	179.00	213.00	289.00	378.00	479.00	591.00
15	34.60	47.25	116.00	147.00	181.00	219.00	260.00	354.00	463.00	585.00	723.00
20	46.20	54.55	134.00	169.00	209.00	253.00	301.00	409.00	535.00	676.00	835.00
25	57.70	61.00	149.00	189.00	234.00	283.00	336.00	458.00	598.00	756.00	934.00
30	69.30	66.85	164.00	207.00	256.00	309.00	368.00	501.00	655.00	828.00	1023.00
35	80.80	72.20	177.00	224.00	277.00	334.00	398.00	541.00	708.00	895.00	1106.00
40	92.40	77.20	188.00	239.00	296.00	357.00	425.00	578.00	756.00	957.00	1182.00
45	103.90	81.80	200.00	253.00	313.00	379.00	451.00	613.00	801.00	1015.00	1252.00
50	115.50	86.25	211.00	267.00	330.00	399.00	475.00	647.00	845.00	1070.00	1320.00
55	127.00	90.40	221.00	280.00	346.00	418.00	498.00	678.00	886.00	1121.00	1385.00
60	138.60	94.50	231.00	293.00	362.00	438.00	521.00	708.00	926.00	1172.00	1447.00
65	150.15	98.30	241.00	305.00	376.00	455.00	542.00	737.00	964.00	1220.00	1506.00
70	161.70	102.10	250.00	317.00	391.00	473.00	563.00	765.00	1001.00	1267.00	1565.00
75	173.20	105.70	259.00	327.00	404.00	489.00	582.00	792.00	1037.00	1310.00	1619.00
80	184.80	109.10	267.00	338.00	418.00	505.00	602.00	818.00	1070.00	1354.00	1672.00
85	196.30	112.50	276.00	349.00	431.00	521.00	620.00	844.00	1103.00	1395.00	1723.00
90	207.90	115.80	284.00	359.00	443.00	536.00	638.00	868.00	1136.00	1436.00	1773.00
95	219.40	119.00	292.00	369.00	456.00	551.00	656.00	892.00	1168.00	1476.00	1824.00
100	230.90	122.00	299.00	378.00	467.00	565.00	672.00	915.00	1196.00	1512.00	1870.00

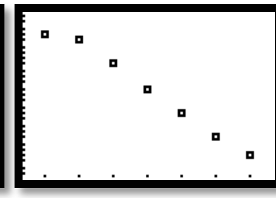
Section 6.2: Resistant Data

The problems in section 6.1 were chosen because in determining which model fit the data best there was one equation that stood out as a model above the others. This section will give you some practice at choosing a model for data that doesn't fit nicely with one of the five equations we studied without modifications.

Two of the simple modifications that can be made are:

1. **Omitting points** - If there is a point that doesn't fit with the data, particularly if it is the first point in a pattern over time, removing it may allow a more accurate model while still being honest with the data. Removing it would be dishonest in this case if your goal was to estimate what the population was before 2002.

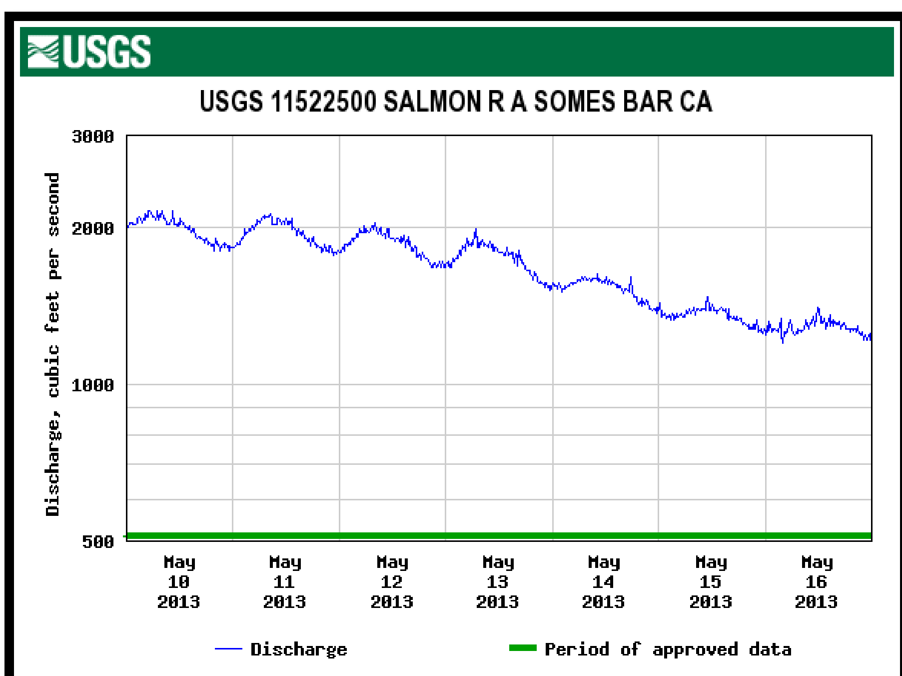




Some regression equations do not allow for zeros in the data:

a) Power and logarithmic regression will not allow a point with an x-coordinate of 0.
b) Power and exponential regression will not allow a point with a y-coordinate of 0.

2. **Different models** - There are other models that we have not covered in this course that may work better. This graph is a sine wave reflecting the effect of the melt/freeze cycle on the Salmon River level during spring; and it is decreasing each day as the snowpack diminishes.



Consider the ballistics data:

Example 6.2.1: The Deflection of a Bullet in the Wind

The chart and graph show how far a bullet is deflected to the left or right of its target in a 90° angle wind. Consider the data for the Ackley Hornet (AH) in a 10 MPH wind.

Dist (Yd)	17 HMR 5 MPH (in)	17 HMR 10 MPH (in)	17 AH 5 MPH (in)	17 AH 10 MPH (in)
0	0.0	0.0	0.0	0.0
25	0.1	0.2	0.0	0.1
50	0.4	0.8	0.2	0.3
75	0.9	1.8	0.3	0.7
100	1.6	3.3	0.6	1.2
125	2.6	5.3	1.0	2.0
150	3.9	7.9	1.4	2.9
175	5.6	11.1	2.0	4.0
200	7.5	15.1	2.6	5.3
225	9.9	19.8	3.4	6.8
250	12.6	25.3	4.3	8.6
275	15.8	31.5	5.3	10.6
300	19.2	38.5	6.4	12.9

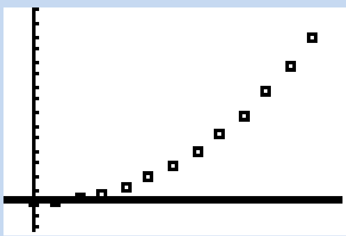
a) Use the features of your graphing calculator to find the best model for the data where the deflection in inches is a function of the distance of the shot in yards.

b) Use the equation to predict the deflection of a 375 yard shot.

c) Use the equation to estimate the distance of a shot that deflects 9 inches.

Solution:

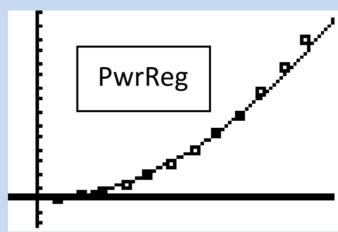
1. **The eyeball test:**

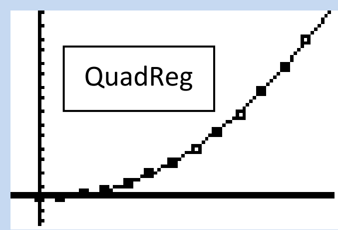


The eyeball test would suggest quadratic, power or exponential as possible options.

2. **R-value:**

Power and exponential regression will not work unless we omit the (0,0) point. Doing so, the power equation has a strong r-value of (.9986) but the graph begins to deviate from the data at the longer distance shots. Quadratic regression yields a very strong r-value of (.9996) without having to remove (0,0) and the graph fits the data almost perfectly.





3. **Common sense:**

The right side of the parabola continues to increase which agrees with the bullet deflecting more in the wind with the distance of the shot.

Final Answer:

QuadReg
y=ax²+bx+c
a=1.584016E-4
b=-.0054105894
c=.1230769231
R²=.9996278519

a) The quadratic equation: $y = .0001584x^2 - .00541x + .1231$ is the best model for the data. Note the scientific notation in "a" (E -4), meaning the decimal point is actually 4 places to the left.

b) Substituting 375 for "x" predicts a deflection of 20.4 inches.

c) Substituting 9 for "y" and using the quadratic formula to solve for "x" yields an estimated shot distance of 254.4 yards. Notice this fits in between the 250 and 275 yard entries in the table.

Consider the river speed example:

Example 6.2.2: River Flow -vs- River Speed

The time it takes the water in the Trinity River in northern California, released from the dam at Lewiston, to reach Cedar Flat is shown in the table.

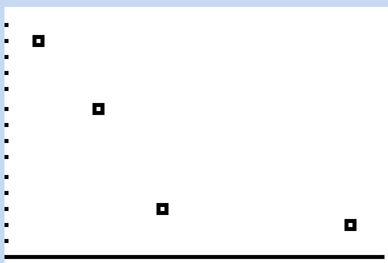
Use the features of your graphing calculator to find the best model for the data. Then use the equation to predict the time to Cedar Flat at 2000 CFS.

Lewiston Release	Time to Cedar Flat
500 cfs	24 hours
1000 cfs	20 hours
1500 cfs	14 hours
3000 cfs	13 hours

www.dreamflows.com/Pages/TrinitySchedule.php

Solution:

1. **The eyeball test:**

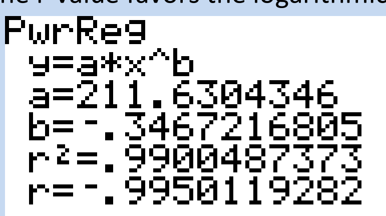


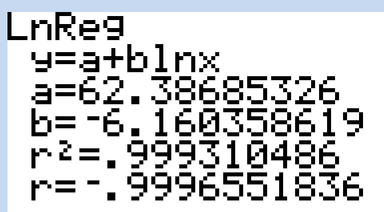
The data actually fails the eyeball test, as it does not appear there is a clear relationship. It certainly makes sense that higher flows will increase the speed of the river and thus decrease the time it will take to travel a distance. This relationship is visible in the table but then defies common sense when viewed as a graph. This author suspects this data is not very accurate … despite the fact that it was on the internet.

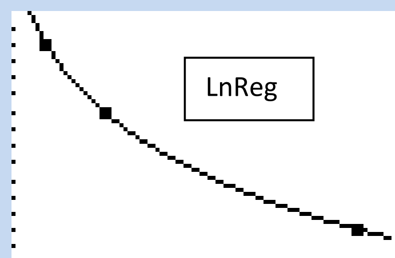
The point that seems to defy the pattern is (1500,14) … let's remove it!

2. **R-value:**

The r-value favors the logarithmic model







3. **Common sense:**

It makes sense that the speed of the river would increase up to some limit, thus allowing the time to decrease to some limit. It also makes sense that there would not be any "jumps" or "gaps" in the data as the original points indicated. Removing the point can be acceptable if you note it in your conclusions. It would of course be ideal to go test your hypothesis by collecting data a little more scientifically accurate.

Final Answer: The logarithmic model is the best if we omit the point (1500,14).
Substituting 2000 for x in the logarithmic equation: $y = 62.387 - 6.160\ln(x)$ and solving for y, gives a time of 15.6 hours.

Section 6.2: Problem Set

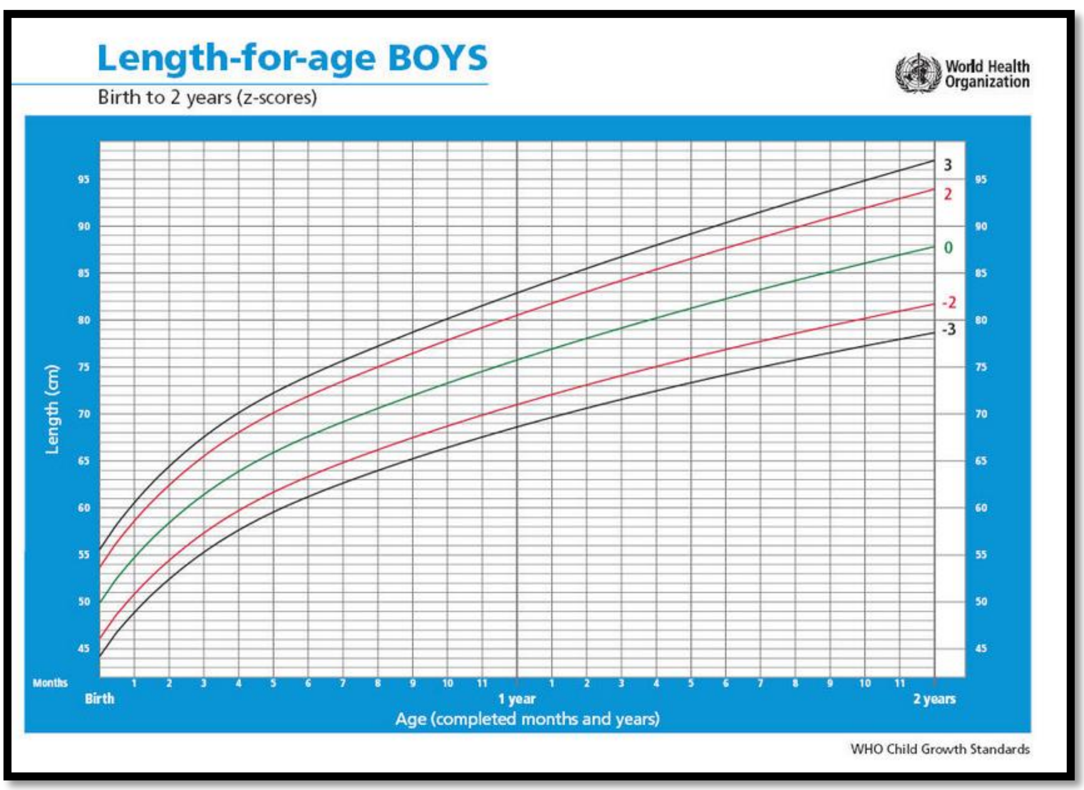
1. The average weight for a girl during the first years of life is shown in the chart.

   a) Use regression and the graph of the data to find the best model for the data.
   b) Briefly defend the equation you chose to model the data.
   c) Use your equation to predict the weight at 24 months.
   d) Use your equation to predict the month a girl would weigh 32 pounds.

Age (months)	Weight (pounds)
birth	7.13
1	9.26
2	11.2
4	14.1
7	16.82
10	18.69
12	19.72
16	21.6

2. The growth curve for boys used by health care professionals is shown below. The center curve represents the average length and the outer curves pertain to 2 and 3 standard deviations from this average (a statistics course will teach you what to make of standard deviation).

   a) Identify some ordered pairs from the center graph and use regression and the graph of the data to find the best model for the data. We did this question in section 6.1, but try choosing length for (x) and age in months for (y). This relationship can be very accurately modeled using this creative approach.
   b) Briefly defend the equation you chose to model the data.
   c) Use your equation to calculate to predict the length a boy "should" be at 36 months.



3. The table shows how barrel length effects muzzle velocity (ft/sec) for a 9mm Luger Federal 115 grain bullet.

   a) Use regression and the graph of the data to find the best model for the data.
   b) Briefly defend the equation you chose to model the data.
   c) Use your equation to predict muzzle velocity for a 24" barrel.
   d) Redo regression but use the logistic model to find a model for the data (at the end of the list of regression equations on your calculator).
   e) Use the logistic model to predict the muzzle velocity for a 24" barrel.

Barrel Length	Federal 115 gr.
18"	1297
17"	1320
16"	1295
15"	1304
14"	1295
13"	1281
12"	1282
11"	1268
10"	1253
9"	1238
8"	1234
7"	1223
6"	1188
5"	1166
4"	1094
3"	1029
2"	948

4. The chart shows the relationship between the number of clicks on a scope and its effect on the drop in a bullet.

   a) Use regression and the graph of the data to find the best model for the data considering number of clicks as a function of the drop.
   b) Briefly defend the equation you chose to model the data.
   c) Use your equation to determine the number of clicks necessary to adjust for a drop of 48 inches.
   d) Use your equation to predict the drop for 250 clicks.

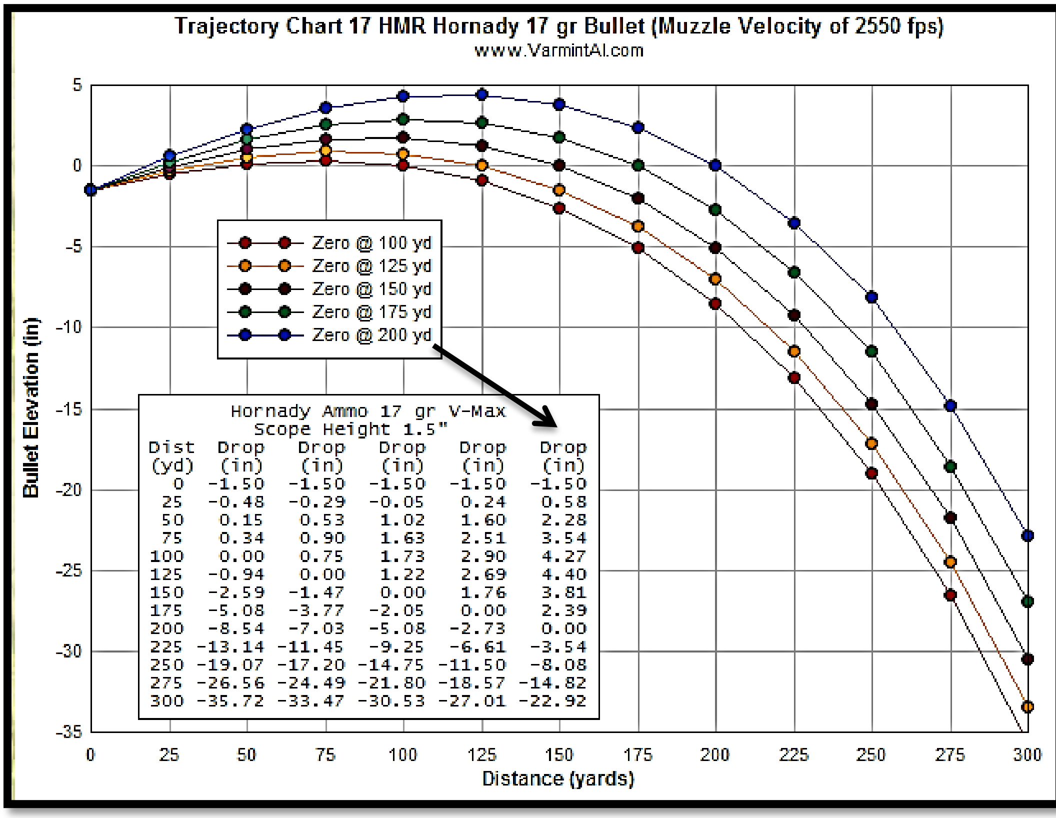
Hornady XTP 20 gr. 17 HMR MV=2375

Click Size 0.125

Yard	Drop	Click
100	0.0	0
125	1.2	8
150	3.2	17
175	6.1	28
200	10.2	41
225	15.3	55
250	22.6	73
275	31.2	91
300	41.7	112
325	54.1	134
350	68.3	157
375	84.9	182
400	103.7	208

5. The chart and graph show how far a 17 HMR Hornady 17 grain bullet will drop as a function of the length of the shot measured in yards.

   a) Use regression to find a **cubic model** for the data for the scope that is zeroed at 200 yards. You will notice this has a remarkably good fit since the data does not have the symmetrical shape that quadratic graphs have.
   b) Use your equation to predict the drop for a 350 yard shot.



Dist (yd)	Drop (in)	Drop (in)	Drop (in)	Drop (in)	Drop (in)
0	-1.50	-1.50	-1.50	-1.50	-1.50
25	-0.48	-0.29	-0.05	0.24	0.58
50	0.15	0.53	1.02	1.60	2.28
75	0.34	0.90	1.63	2.51	3.54
100	0.00	0.75	1.73	2.90	4.27
125	-0.94	0.00	1.22	2.69	4.40
150	-2.59	-1.47	0.00	1.76	3.81
175	-5.08	-3.77	-2.05	0.00	2.39
200	-8.54	-7.03	-5.08	-2.73	0.00
225	-13.14	-11.45	-9.25	-6.61	-3.54
250	-19.07	-17.20	-14.75	-11.50	-8.08
275	-26.56	-24.49	-21.80	-18.57	-14.82
300	-35.72	-33.47	-30.53	-27.01	-22.92

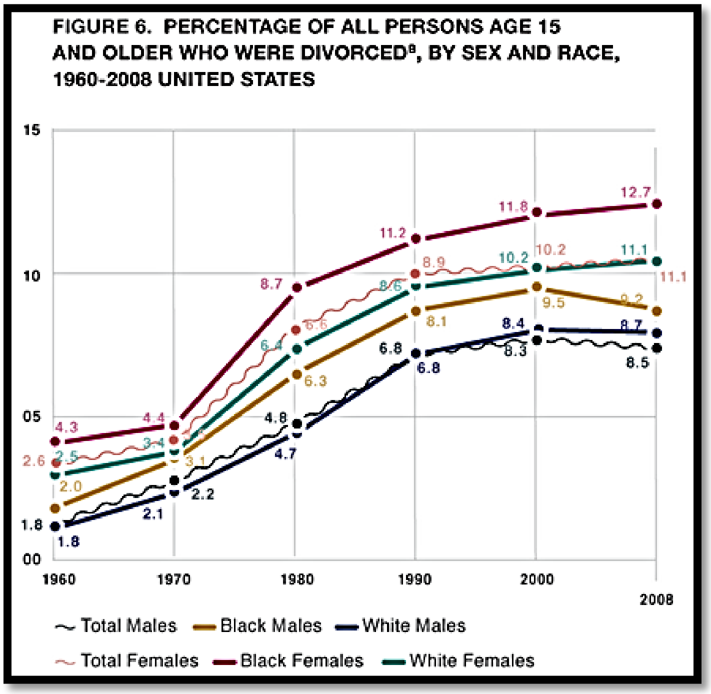
6. The percent of divorced white females is shown in the table and graph since the 1960's.

Year	Percent
1960	2.5
1970	3.4
1980	6.4
1990	8.6
2000	10.2
2008	11.1

a) Use regression and the graph of the data to find the best model for the divorce percent for white females (the bottom graph)

b) Briefly defend the equation you chose to model the data.

c) Use your equation to predict the percent you would expect in 2020.



7. The chart shows the number of clicks on the scope that are necessary to adjust for the drop in a bullet for different length shots.

a) Use regression and the graph of the data to find the best model for the data considering number of clicks as a function of yardage.

b) Briefly defend the equation you chose to model the data.

c) Use your equation to determine the number of clicks necessary for a 190 yard shot.

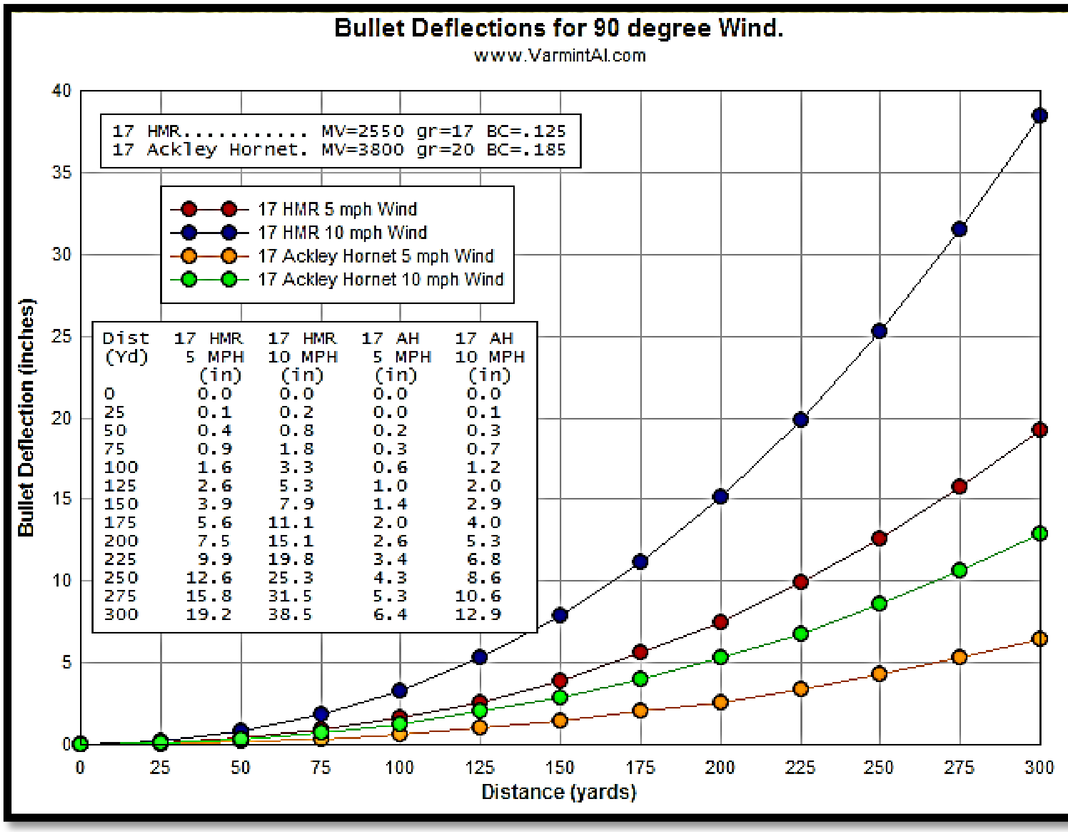
Hornady XTP 20 gr. 17 HMR MV=2375

Click Size 0.125

Yard	Drop	Click
100	0.0	0
125	1.2	8
150	3.2	17
175	6.1	28
200	10.2	41
225	15.3	55
250	22.6	73
275	31.2	91
300	41.7	112
325	54.1	134
350	68.3	157
375	84.9	182
400	103.7	208

8. The table and graph show how far a bullet is deflected to the left or right of its target in a 90° angle wind. Consider the data for the 17 HMR in a 10 MPH wind as a function of the distance.

   a) Use regression and the graph of the data to find the best model for the data.
   b) Briefly defend the equation you chose to model the data.
   c) Use your equation to determine the deflection of the bullet for a 350 yard shot.
   d) Use your equation to estimate the yardage for a shot that is deflected 17 inches.



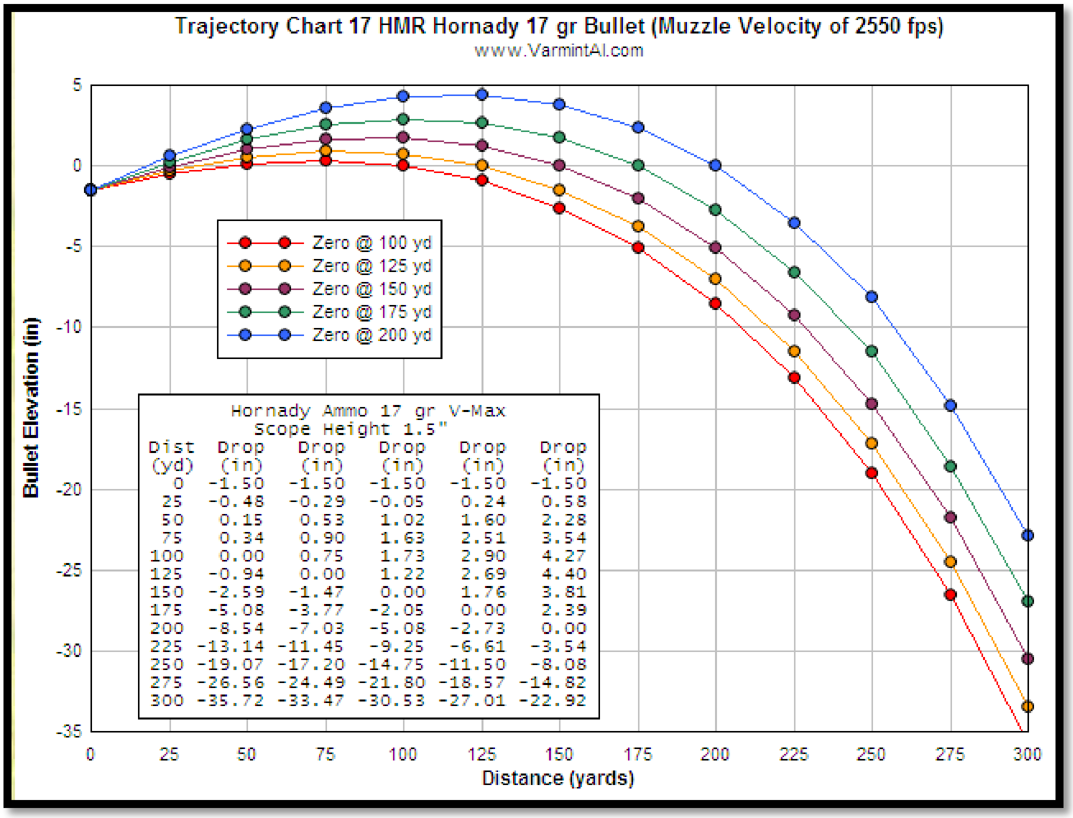
9. The table is used to size home water systems. It shows the feet of head (height of the source above the faucet) that are necessary for different pounds per square inch (PSI) of pressure at the faucet.

   a) Use regression to find the best model where the pressure is a function of feet of head.
   b) Briefly defend the equation you chose to model the data.
   c) Use your equation to predict the feet of head that will be necessary to achieve 80 pounds of pressure.

Head (feet)	Pressure (PSI)
2.31	1
4.62	2
6.93	3
9.24	4
11.6	5
13.9	6
16.2	7
18.5	8
20.8	9
23.1	10
25.4	11
27.7	12

10. The chart and graph show how far a 17 HMR Hornady 17 grain bullet will drop as a function of the distance of the shot measured in yards.

    a) Use regression to find the best model for the data for the scope that is zeroed at 100 yards.
    b) Briefly defend the equation you chose to model the data.
    c) Use your equation to predict the drop for a 350 yard shot.



Dist (yd)	Drop (in)	Drop (in)	Drop (in)	Drop (in)	Drop (in)
0	-1.50	-1.50	-1.50	-1.50	-1.50
25	-0.48	-0.29	-0.05	0.24	0.58
50	0.15	0.53	1.02	1.60	2.28
75	0.34	0.90	1.63	2.51	3.54
100	0.00	0.75	1.73	2.90	4.27
125	-0.94	0.00	1.22	2.69	4.40
150	-2.59	-1.47	0.00	1.76	3.81
175	-5.08	-3.77	-2.05	0.00	2.39
200	-8.54	-7.03	-5.08	-2.73	0.00
225	-13.14	-11.45	-9.25	-6.61	-3.54
250	-19.07	-17.20	-14.75	-11.50	-8.08
275	-26.56	-24.49	-21.80	-18.57	-14.82
300	-35.72	-33.47	-30.53	-27.01	-22.92

Citations

Chapter 1: Linear Relationships

Homework

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1.1.2: Tree Dia./Vol.: http://treephys.oxfordjournals.org/content/early/2013/01/15/treephys.tps127/suppl/DC1
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1.1.5: Olympic Discus: http://www.statsci.org/data/general/olympic.html
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1.1.8: Upload Speed: http://datamarket.com/en/data/set/1gyb/#!ds=1gyb!1c89=4j:1c8a&display=line
1.1.9: Adult Smokers: http://www.cdc.gov/tobacco/data_statistics/tables/trends/cig_smoking/index.htm
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1.3.3: Coffee Prices: http://thehumanbean.com/menu/
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Chapter 2: Quadratic Relationships

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2.2.10: Tree H/V: http://link.springer.com/article/10.1007%2Fs11676-014-0427-4#page-1

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Chapter 3: Power Relationships

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3.1.3: Flow Rate: http://www.industrial-equipment.biz/Information/gas-piping.html#NatGas

3.1.4: Collisions at RR crossings: http://oli.org/about-us/news/collisions-casulties

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Chapter 4: Exponential Relationships

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Chapter 6: Finding the Right Model

Homework

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